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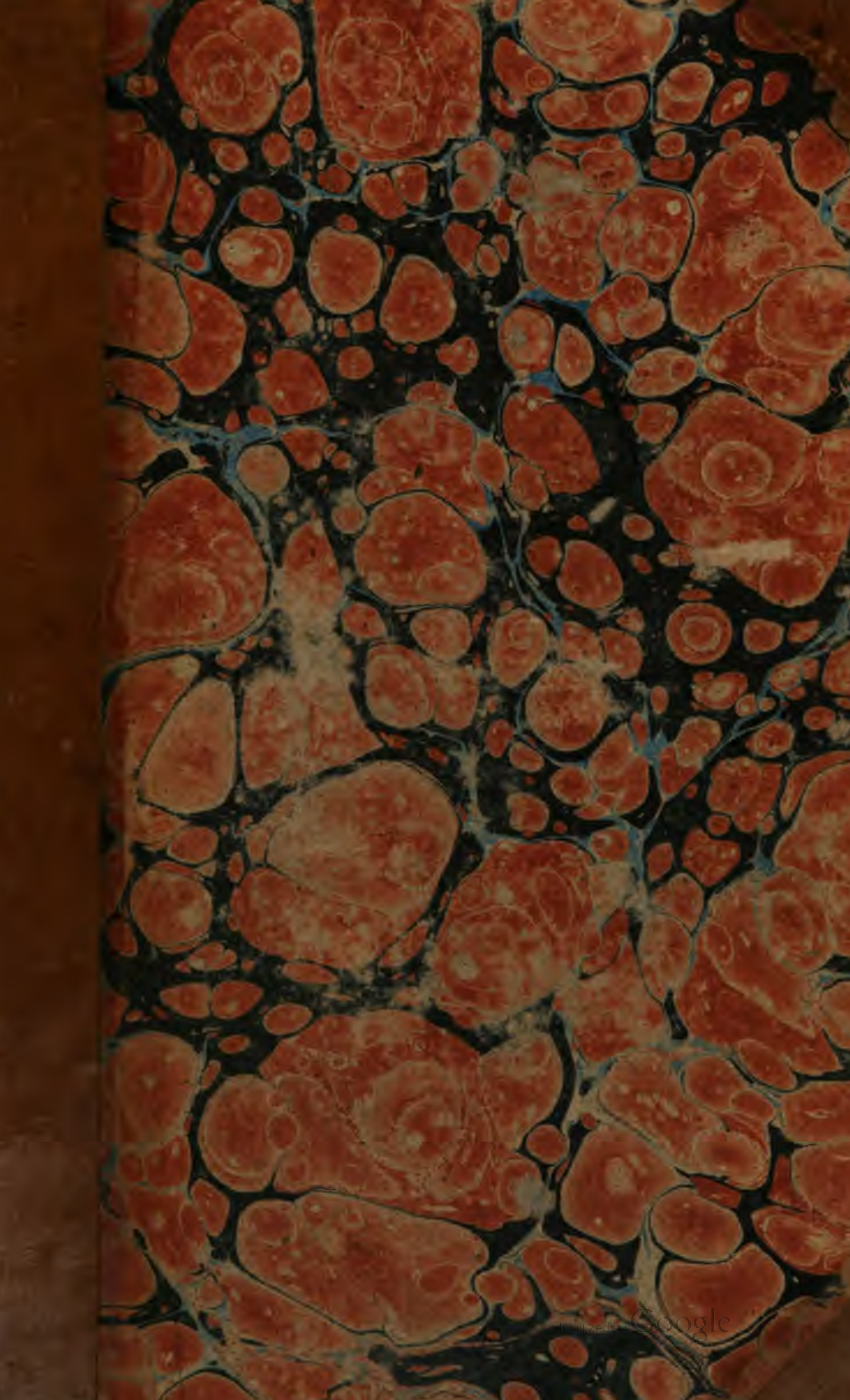

OUTLINES

OF

URAL PHILOSOPHY.

Alexr. Marcet

BCU - Lausanne

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OUTLINES

OF

NATURAL PHILOSOPHY.

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OF

NATURAL PHILOSOPHY,

BEING

HEADS OF LECTURES

DELIVERED IN THE UNIVERSITY OF EDINBURGH,

BY

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CORRIGENDA.

Page 28, line 8 from the bottom, *for* FAC, *read* FAB,
3 from do. *for* the other *read* their
77, art. 136, *for* (fig. 10.) *read* (fig. 10. *)
87, 15 from the top, for *Vie* read *Vis*
127, line 13 from the top, *for* D, *read* P,
146, 10, *for* (fig. 13.) *read* (fig. 18.)
155, 20, *for* (fig. 12.) *read* (fig. 19.)
28, *for* CF, *read* CE,
173, art. 262, W is the weight of the floating body,
187, line 3, for g, read q,

OUTLINES

OF

NATURAL PHILOSOPHY.

INTRODUCTION.

SECT. I.

1. PHILOSOPHY is the knowledge of the general laws observed by the phenomena of nature, whether in the intellectual or material world.

2. In the material world, the action which takes place among bodies, either produces a permanent change in the internal constitution of those bodies, or it does not. In the former case, the phenomena belong to CHEMISTRY, in the latter to NATURAL PHILOSOPHY.

The action of the same cause may, by a change of circumstances, pass from being an object of the one of these sciences to become that of the other. The action of Heat, when it expands or contracts the dimensions of bodies, belongs to Natural Philosophy;

when the same power burns and consumes bodies, its action belongs to Chemistry. When it converts water into steam it may belong to either science.

3. When from a comparison of a number of facts known from experiment or observation to be true, the existence of a more general fact is inferred, the inference is said to be made by INDUCTION.

It is from induction that all certain and accurate knowledge of the laws of nature is derived.

4. When general principles or axioms have been established by induction, we can often, by the application of mathematical reasoning, deduce conclusions as clear and certain as the principles themselves.

The great advantage which Natural Philosophy seems here to possess exclusively, arises from this, that the action which it treats of, extends to large masses of matter, and to considerable distances, such as can be measured by lines and numbers.

5. The branch of knowledge which collects and classifies facts, is called Natural History.

Natural History is here taken in its most general sense, such as it is understood by BACON; in common language its objects are confined to what are called the three kingdoms, the Mineral, Vegetable, and Animal.

6. We are said to *explain* a phenomenon, when

we shew it to be necessarily included in some phenomenon or fact already known, or supposed to be known; and we consider one phenomenon as the cause of another, when we conceive the existence of the latter to depend on some force or power residing in the former.

7. A fact assumed in order to explain appearances, and having no other evidence of its reality, but the explanation it is supposed to afford, is called a HYPOTHESIS.

8. An explanation of any system of appearances or events founded on facts known to exist from evidence independent of those appearances, is called a THEORY.

In the explanation of natural appearances, and in all inductive reasonings, Facts, though equally certain, may be of different value for the discovery of truth. BACON has classified facts, and explained their peculiar advantages as instruments of investigation, in the 2d book of the NOVUM ORGANUM.

9. When one system of events or appearances is similar to another, and when we infer that the causes in these two systems are also similar, we are said to reason from ANALOGY.

Such reasonings will be more or less conclusive, according as the similarity is more or less considerable. Explanations, therefore, or theories founded on analogy, may have all degrees of evidence, from the least to the greatest.

Many theories are founded entirely on analogy.

10. When a theory has been discovered by induction, it may be made use of by reasoning in a reverse order, for discovering new facts, and predicting the result of new combinations.

This will be best illustrated by the examples that will occur in treating of Physical Astronomy.

11. The evidence of a theory, increases with the number of facts which it explains, and the precision with which it explains them. It diminishes with the number of facts which it does not explain, and with the number of different suppositions that will afford explanations equally precise.

In the analytical or inductive method of Bacon, the possible theories are all excluded but one or at most a small number, before the explanation is attempted. *Novum Organum*, Lib. II. cap. 16.

A theory may not deserve to be rejected, though it does not explain all the phenomena, if it explain a great number, and be not absolutely inconsistent with any one. A single fact, inconsistent with a theory, may be sufficient to overturn it.

12. In tracing the necessary connection of natural phenomena, this axiom is often of use: Nothing exists in any state that is not determined by

some REASON to be in that state rather than in any other.

Hence two things of which the conditions are determined by reasons that are precisely the same, are in all respects similar to one another. Hence, also, if there are two conditions, and no reason to determine a subject to be in one of them rather than another, we are to conclude that it is in neither. This axiom has been called the principle of the SUFFICIENT REASON. It was used by ARCHIMEDES in his demonstration of some propositions in mechanics; but it was first stated as a general principle of philosophic reasoning by LEIBNITZ. It may be used to great advantage for demonstrating the more simple propositions of geometry as well as of mechanics.

13. Experiment is not only necessary in the investigation of truth, but it is useful for proving truths that have already been investigated.

In this latter application of it, it serves three purposes:

I. It verifies the conclusion of our reasonings, and discovers whether any element has been left out, or any error committed in the course of the investigation.

II. It exemplifies the application of general principles to the explanation of particular facts.

III. It impresses the truth both of the principles and the conclusion most strongly on the mind.

Though an experiment can only prove a proposition in one particular case, yet from a combination of two or more experiments, such evidence of the general proposition may arise, as to fall little short of demonstration.

14. The study of Natural Philosophy is accompanied with great advantages.

I. It extends man's power over nature, by explaining the principles of the various arts which he practises.

II. It improves and elevates the mind, by unfolding to it the magnificence, the order, and the beauty manifested in the construction of the material world.

III. It offers the most striking proofs of the beneficence, the wisdom, and the power of the CREATOR.

SECT. II.

PROPERTIES OF MATTER.

15. BODY is a substance, extended in three dimensions, impenetrable, and moveable.

Gravity or weight is often included in the description of body; it is left out here, because though it appear to be universal, we can conceive a substance without gravity, which, nevertheless, should possess extension, impenetrability, and mobility.

16. By the Impenetrability of body is meant, that no two particles of matter can at the same time occupy the same identical portion of space.

If bodies could compenetrate, all matter might be united into any space, however small, or it might be all annihilated.—MUSCHENBROEK, *Introd. ad Phil. Nat.* vol. i. § 81.

The bodies that yield to pressure, and those that do not, serve equally to prove the impenetrability of matter.—BOSCOVICH, *Theoria Phil. Nat.* § 41.

17. Body, being extended, is divisible without limit, or, as it is usually called, *ad infinitum*, supposing that the instruments of division are mere mathematical lines or points.

It is easily proved, on the supposition just mentioned, that the extension of lines *ad infinitum*, and their divisibility *ad infinitum*, are necessarily connected with one another.

18. The actual division of body into parts, as the parts must be of a finite magnitude, must necessarily be limited; it is, nevertheless, capable of being carried to such extent, that the parts shall be of incredible minuteness.

We have examples in the gilding of silver-wire, in the propagation of odours, in the colours produced by chemical solutions, &c.—REAUMUR, *Mém. Acad. des Sc.* 1713, p. 204. BOYLE *de mirâ Subtilitate Effluviorum*. MUSCHENBROEK, *ubi supra*, § 72.

19. All bodies are inclosed by one or more boundaries, and therefore possess figure; they have also the capacity of receiving an indefinite variety of figures.

Bodies differ in their capacity for receiving and maintaining different figures.

Some receive new figures with difficulty, but maintain them easily. Such are the bodies usually called *solid.*

Others receive any figure easily, but cannot maintain it without the assistance of other bodies. Fluid bodies are of this kind.

Many bodies have figures that in their natural state are peculiar to them. This is the case not only with plants and animals, but with certain mineral and chemical productions called Crystals, usually bounded by plain surfaces.

20. Motion, of which all bodies are susceptible, is the continued change of place.

21. It is a general law in the material world, that no body loses motion in any direction, without communicating an equal quantity to other bodies in that same direction; and conversely, that no body acquires motion in any direction, without diminishing the motion of other bodies by an equal quantity in that same direction.

What relates to the motion of bodies, will be considered more fully hereafter; it is mentioned here merely for the sake of order, as being part of the definition of body.

22. The force with which the parts of bodies resist any endeavour to separate them is called *cohesion*.

We may conceive a body to be made up of an assemblage of small indivisible particles or corpuscles adhering to one another, with forces that are greater as the distance of the particles is less. These corpuscles are called the *elements of body*.

Hardness, softness, tenacity, fluidity, ductility, are modifications of cohesion.

Smooth surfaces may be made to cohere, by bringing them very close to one another; but,

In general, cohesion cannot be produced between two bodies but by the intervention of fluidity.

Cohesion is not a general property of body; it does not belong to elastic fluids, which are kept together by pressure or by gravity.

23. All bodies at the earth's surface, if left to themselves, descend in straight lines towards that surface. This tendency or disposition to fall is called *weight*.

The weight of a given body may be employed to measure the weights of all other bodies.

24. The directions in which bodies fall in

different places of the earth, tend nearly to the centre of the earth.

At points not very far from one another on the surface, the directions in which bodies gravitate are nearly parallel, the distance of the earth's centre being great in respect of the distance of the bodies from one another.

If the distance is a nautical mile, or 6118 feet nearly, the angle made by the directions of gravity is one minute; if 60 of those miles, it is a degree, &c.

A plane at any place perpendicular to the line in which bodies gravitate, is called a *horizontal plane*; and any plane passing through that line is called a *vertical plane*.

25. If we conceive the weight of bodies to be produced by a force applied to each of their elementary particles, urging it downwards, this force is called *gravity*.

Gravity must be distinguished from Weight: the weight of a body is the product of the gravity of a single particle by the number of particles. Thus,

If W be the weight of a body, g the gravity of a single particle, N the number of particles, $W = N \times g$.

26. In all bodies the force of gravity is the same.

This is proved by the fact, that all bodies, whatever be their weight, fall at the same rate to the ground, when the resistance of the air is removed.

27. The weights of two bodies are to one another as the quantities of matter in those bodies.

Let W and W′ be the weights of two bodies M and M′, their quantities of matter,

$$\frac{W}{W'} = \frac{M}{M'}.$$

The quantity of matter in a body is called its *mass*.

28. All bodies have empty spaces disseminated through them in the form of *pores*, more or less minute. This property of bodies is called *porosity*.

In solids the pores can be seen sometimes by the naked eye, and almost always by the microscope.

In fluids the dissemination of vacuity through the mass is proved by experiments, though it is not perceived by the eye.—Muschenbroek, § 91. NO. 3.

The ratio of the quantity of matter to the quantity of empty space contained within the superficies of any body is wholly unknown.

It is probable, that even in the densest bodies, the quantity of solid matter is very small, compared with the quantity of empty space. Newton's *Optics*, Book II. Part iii. Prop. 8.

From the porosity of bodies, it follows, that the particles of matter can only touch one another in a few points.

29. If in two bodies, of which the bulks are B and B′, and the weights W and W′, the weights be proportional to the bulks, or $\frac{W'}{W} = \frac{B'}{B}$, the space occupied by the pores of these bodies bears the same proportion to that which is filled by solid matter.

Bodies of this kind are said to have the same density.

30. The weight of a body divided by its quantity of matter, or $\frac{W}{B}$, is called the *specific gravity* of the body.

The weight of a body, in strictness, cannot be divided by its bulk, the two quantities being dissimilar, and incapable of comparison. But if we express the bulk of all bodies numerically, by referring them to the same unit, a cubic inch for example; and the weight of all bodies, by referring them to a similar unit, the weight of a cubic inch of water, any number in the first series may be divided by the corresponding number in the second: the quotient will be the specific gravity of the body to which the numbers belong.

Hence the specific gravity multiplied by the bulk, gives the weight; or if S is the specific gravity, $B \times S = W$.

31. A fluid is a body so constituted, that its parts are all ready to yield to the action of the least pressure.

32. *Elasticity* is a power by which a body, when its figure is altered by the action of any force, resumes that figure as soon as the force has ceased to act.

This power is found both in solid and fluid bodies.

33. All bodies are subject, by expansion and contraction, to change their magnitude or volume according to certain laws, and within certain limits. When expanding, they produce in us (under certain conditions) the sensation of heat; and when contracting, the sensation of cold. They are therefore said to become hot, or to acquire *heat* in the former case, and to become cold or to lose *heat* in the latter.

34. Certain minerals have a power of attracting iron, and of communicating to pieces of that metal the power of attracting and repelling one another. The power thus communicated is called *magnetism*. Bodies having magnetism have also a tendency, when left free, to point toward a certain quarter of the heavens. This is called *polarity*.

Magnetism is a permanent quality; it is peculiar to iron and its ores.

35. Some bodies acquire by friction the power of attracting and repelling certain other bodies. This power is called *electricity*.

36. The chemical action of metallic substances on one another, produces a like tendency in bodies communicating with those substances according to a certain rule. This is called *galvanism*.

The four properties or powers last mentioned agree in this peculiarity, that they are communicated from one body to another by mere apposition without any visible admixture, or transfusion of substance. In this they differ from all the properties of matter yet known. Whether a passage of some invisible substance from the one body to the other is not to be inferred in such instances, will hereafter be considered.

The properties of matter above enumerated, require the science which treats of body to be divided into different branches. As the action of bodies on one another generally involves *motion*, the consideration of that power constitutes one of the main objects of natural philosophy. The doctrine which treats of it is, however, modified by the nature and constitution of the bodies to which it is communicated.

37. When bodies are free to obey the impulses communicated to them, the science which treats of their motion is called DYNAMICS.

Dynamics is the most elementary branch of the doctrine of motion, and the most general in its principles.

As there is here no question concerning the figure of the moving bodies, we may abstract entirely from the consideration of it, and treat of the bodies as if their matter were concentrated in mere physical points. The term Dynamics signifies literally the doctrine of *power*, power or force being known to us only as the cause of motion, and measured by the motion it produces,

38. When bodies, whether by external circumstances, or by their connection with one another, are not left at liberty to obey the impulses given, the principles of dynamics must receive a certain modification before they can be applied to them. The science of dynamics, thus modified, is called MECHANICS.

According to this distinction, the motion of a body falling freely to the ground belongs to Dynamics; the motion of the same body descending on an inclined plane belongs to Mechanics, &c.

The doctrine of Machines belongs to mechanics; for in every machine, the connection of the parts prevents them from immediately obeying the impulses received.

39. When the bodies to which motion is communicated are fluid, another modification of the principles of dynamics takes place, which constitutes the science of HYDRODYNAMICS.

In each of these three sciences, there are cases in which the forces conceived to act balance one another, and produce not motion, but rest. These cases in dynamics and mechanics constitute a particular branch, to which the name of *statics* has been given. The similar cases in fluids form the branch of hydrodynamics, called *hydrostatics*.

Hydrodynamics is also subdivided, according as the fluids are incompressible, like water, or elastic, like air.

DYNAMICS.

DYNAMICS.

SECT. I.

OF THE MEASURES OF MOTION.

40. WHEN a body changes its place continually, it is said to MOVE, or to be in MOTION.

41. A moving body cannot pass from one point of space to another without passing through every intermediate point; that is, whatever be the path it describes, it must intersect all the planes that can pass any how in the interval between the two points.

42. The ideas of Space and Time are both necessarily involved in the idea of motion.

43. Time is conceived as a quantity consisting of

consisting of parts which can be compared with one another.

44. Two events which are determined by circumstances precisely the same, are conceived to happen in equal portions of time.

A stone will fall to the ground from a given height in the same space of time to-day that it did yesterday, or that it will do to-morrow, or at any future period. It is not necessary that the stone should be of the same weight, for, as has been already proved, the weight of the falling body does not affect the time of its descent (26). It is on this proposition, generalized and applied to the vibrations of a pendulum or a balance, that the going of a clock or watch is taken for a measure of time.

Thus it is on the principle of the *sufficient reason* that time is divided into equal portions.

The imperfections of mechanism require, however, that our chronometers should be frequently checked by a comparison with the motion of the heavenly bodies. Here also we proceed on the principle just mentioned; but as the circumstances that may influence the duration of the events are in this case not easily discovered, it has required the highest improvements in science to derive an exact measure of time from astronomical observation.

45. *Absolute* time is that which is computed

from the same moment; *relative* time is that which is computed from different moments.

> The least portion of time that we can measure is about one-fourth or one-fifth of a second; yet, were we to trace the progress of nature minutely, a hundred or even a thousandth part of a second would be found distinguishable by great changes.

46. The velocity of a moving body, or the rate of its motion, is said to be *uniform*, when the lines it passes over (or, as it is usually expressed, the spaces it describes) in equal times, are all equal to one another.

> In measuring the velocities of bodies, it is convenient to take a certain portion of time for the unit, in terms of which all other portions of time are to be expressed. The unit here assumed is one second.

47. The space which a body moving with a uniform velocity passes over in any time, is had by multiplying the time into the velocity, that is, by multiplying the number of seconds the motion has continued into the space moved over in one second.

> Hence if S be the space passed over in the time T, with the uniform velocity V, (all of them being expressed numerically),

$$S = V \times T.$$

48. In uniform motions, therefore, the velocity is equal to the space divided by the time, or $V = \frac{S}{T}$.

49. In uniform motions also, the time is equal to the space divided by the velocity, or $T = \frac{S}{V}$.

In all these theorems T is expressed in seconds, and V in some known measure of length, the same in which S is also expressed.

50. The velocity of a body is sometimes estimated in a direction different from that in which it actually moves.

Thus, (fig. 1.) if while a body B moves along the line AC, a perpendicular BD be continually drawn to the line AE, which makes with AC a given angle, the velocity with which D moves along AE is called the velocity of B in the direction AE.

51. The velocity of a body moving in a given direction, is to its velocity estimated in any other direction, as radius to the cosine of the angle which the two directions make with one another; or if V be the velocity of the body, β the angle which the path of the body makes with another line, $V \cos \beta$ is the velocity reduced to the direction of that line.

If S be the space described by the body in its own path, the space described by the perpendicular in the other line will be S cos β.

Hence also if the velocity of the body in its own path is uniform, it will be uniform when reduced to any other direction.

52. The change which any variable quantity undergoes in an infinitely small portion of time, is called the Momentary Increment of that quantity.

Thus if S is the space described by a moving body in the time *t*, and if V be its velocity, supposed variable, at the end of the time *t*, if we suppose *t* to be increased by an indefinitely small instant, the change of S and of V in that instant are called their Momentary Increments.

The increments are denoted by the same letters that express the variable quantities, with a point over them. Thus $\dot{S}$, $\dot{V}$, $\dot{T}$, are the momentary increments of S, V and T.

Though the variable quantities may decrease as well as increase, the momentary change is called an Increment in both cases, but is accounted negative when the quantity diminishes.

53. Though the velocity of a body be variable, it may at any point be taken as uniform for an indefinitely small portion of time, and the increment of the space will be, just as in the case of uniform

motion (47.), equal to the velocity multiplied into the indefinitely small portion of time, during which the motion is supposed uniform. Thus, if $\dot{S}$ be the increment of the space, $\dot{t}$ the corresponding increment of the time, and V the velocity,

$$\dot{S} = V\dot{t}\text{; hence}$$

$$V = \frac{\dot{S}}{\dot{t}},$$

$$\dot{t} = \frac{\dot{S}}{V}.$$

These formulæ apply to motion of every kind, whether accelerated or retarded, and express generally the relation between the velocity, the increment of the space, and the increment of the time.

54. The quantity of motion in a moving body is estimated by the product of the mass, or quantity of matter, multiplied by the velocity.

The quantities of motion in two bodies are the same when their velocities are inversely as their masses.

If M and M′ be the masses of two bodies, V and V′ their velocities; if $M : M' :: V' : V$, the quantities of motion are equal, for $M . V = M' . V'$.

55. When one body changes its place relatively to another, it is said to have a relative motion with respect to that other body.

Two bodies if they move with equal velocities toward the same side, in the same or in parallel lines, will have no motion relatively to one another.

If two bodies move with equal velocities in the same straight line, but in opposite directions, their relative motion is the sum of their real motions; if in the same direction, it is the difference of their real motions. The same is to be said of their relative velocities. The use of the signs + and — brings the two parts of this and all similar propositions under one enunciation.

56. The path of one moving body relatively to another is determined, by supposing the latter to stand still in any point of its path, and by inquiting in what line the other must move, and with what velocity, so that it may approach to the former (supposed at rest) or recede from it, at the same rate, and in the same direction, that it actually does when they are both in motion.

SECT. II.

FIRST LAW OF MOTION.

57. A BODY must continue for ever in a state of rest, or in a state of uniform and rectilineal motion, if it be not disturbed by the action of some external cause.

If the body is at rest, it must remain at rest; for if there is no action of another body, there can be nothing to determine it to move in one direction more than in another.

If the body is in motion, it will continue to move in the same direction; for there is nothing to determine its deflections to be in one line more than in another line.

Lastly, it cannot change its velocity; for if its velocity change, that change must be according to some function of the time; so that if C be the velocity which the body has at any instant, and t the time counted from that instant, V the velocity at the end of the time t, the relation between V, C, and t must be expressed thus, $V = C + A t^m + B t^n +$, &c. Now, there is no condition involved, in the nature of the case, by which the coefficients A, B, &c. can be determined to be of any one magnitude rather than of any other; each of them is therefore equal to 0, and the equation is $V = C$, so that the velocity remains constant.

D'Alembert has given a different demonstration of this, *Dynamique*, chap. 1re. See also another by Euler, *Mechanica*, *tom.* 1. § 63.

This proposition is the first law of motion, and, together with the proposition, § 21. constitutes what is termed the inertia or inactivity of matter.

The inertia has been considered by some philosophers as merely the expression of a law of human thought, by which we are determined, when we see a change, to infer the action of a cause. This is inaccurate. That a change never happens in the motion of any body, without an equal and opposite change in the motion of some other body, is as much a fact, independent of the mind that perceives it, as impenetrability, cohesion, or any other phenomenon of the material world.

If a body move in a curve, the continued action of an external force must be inferred: if that action were to cease at any point, the body would continue its motion in a straight line touching its curvilineal path in that point.

58. The cause of motion is denominated Force, and a force is always measured by the effect, that

is, by the motion which it produces in a given time.

> **Two forces are said to be represented by two lines, when the motions which they singly produce are in the directions of those lines, and proportional to them.**

59. Let a body at A (fig. 2.) be acted on by two forces at the same instant, one of which acting alone, would cause it to move uniformly over AB, in a given time; and the other acting alone, would cause it to move over AC, at right angles to AB, in the same time. The velocity of the body in the one of these directions, will not be changed by the force impelling it in the other.

> **The velocity of a body in the direction of a line different from its own path, is measured as explained (art. 51.)**

> **The demonstration of the preceding proposition is founded on the principle of the sufficient reason.**

60. If the lines which each of two forces, acting singly, would have caused a body to describe in the same time, be at right angles to one another, the line which it will describe in that time, when both the forces act on it at once, is the diagonal of the rectangle under the two first-mentioned lines.

61. If the lines which each of two forces acting singly would have caused a body to describe in the same time, make any angle whatsoever with one another, the line which the body will describe in that time, when both the forces act on it at the same instant, is the diagonal of the parallelogram under the two first-mentioned lines.

> This is the celebrated theorem known by the name of the COMPOSITION OF FORCES. The most remarkable demonstrations of it are by DAN. BERNOULLI, *Comment. Petrop.* tom. 1.; D'ALEMBERT, *Opuscules*, tom. 1. *cinquième memoire;* LA PLACE, *Mechanique celeste*, tom. 1. § 1.

62. If two forces be represented by the lines a and b, which contain an angle $= z$, the force compounded from them will be $\sqrt{a^2 + 2\,a\,b\cos z + b^2}$: for this is the diagonal which, in a parallelogram having the sides a and b, and the contained angle z, subtends the supplement of z. Also the sine of the angle which this diagonal makes with a, is $= \frac{a \sin z}{d}$, d being put for the diagonal, or $\sqrt{a^2 + 2\,a\,b\cos + b^2}$.

63. If three forces which impel a body at the same time, be proportional and parallel to the three sides of a triangle, so that the angle contained by any two of them is the supplement of the angle contained by the corresponding sides of the triangle, these forces will be in equilibrium.

> The converse of this is also true, viz. that if the three forces are in equilibrium, they are proportional to the sides of a triangle, &c.
>
> When three forces are in equilibrium, any two of them is greater than the third.
>
> If three forces are in equilibrium, they are all in the same plane.

64. If there be an equilibrium between three forces, AB, AC, AE, (fig. 3.) and if from any point F, in the direction of one of them, perpendiculars FG, FH, be drawn to the other two, these perpendiculars will be inversely as the forces AC, AB, on the directions of which they fall.

> For if FC, FB are drawn, the triangles FAC, FAC on the same base AF, and having equal altitudes, are equal. Therefore $FG \times AC = FH \times AB$, or $FG : FH :: AB : AC$.

65. If three forces are in equilibrium, and if perpendiculars be drawn to the other directions, intersecting one another in three points, the sides of the triangle so formed will be proportional to

the forces, to the directions of which they are perpendicular.

66. If any number of forces act on a body, proportional to the sides of a rectilineal figure, and also parallel to them, in such a manner that the angle which the direction of each force makes with that of the contiguous force, be the supplement of the angle which the corresponding sides of the figure make with one another, these forces will be in equilibrium.

This proposition is true, whether the sides of the rectilineal figure be all in one plane or not. If they inclose a space, that space may be extended either in two or in three dimensions.

67. If there be an equilibrium among any number of forces, which are in different planes, but applied to the same point; if, through that point, three straight lines, or axes, be drawn at right angles to one another, (one of them in a plane perpendicular to that of the other two), and if every force be resolved into parts in the directions of the three axes, then shall the sums of the opposite forces, in the direction of each axis, be equal to one another.

This theorem furnishes three independent equations.

If F, F', F'', F''', &c. be the forces; a, a', a'', a''', &c. the angles which they make with one of the three axes;

axes; $\beta, \beta', \beta'', \beta'''$, &c. the angles which they make with another of the axes, and $\gamma, \gamma', \gamma'', \gamma'''$, &c. the angle which they make with the third,

$$F \cos a + F' \cos a' + F'' \cos a'' + F''' \cos a''' +, \&c. = 0$$

$$F \cos \beta + F' \cos \beta' + F'' \cos \beta'' + F''' \cos \beta''' +, \&c. = 0$$

$$F \cos \gamma + F' \cos \gamma' + F'' \cos \gamma'' + F''' \cos \gamma''' +, \&c. = 0$$

When all the forces are in one plane, there are only two axes, and the equations are therefore reduced to the two first. Also as β is in this case the complement of a, $\cos \beta = \sin a$, and so of the rest; therefore the equations are

$$F \cos a + F' \cos a' + F'' \cos a'' +, \&c. = 0$$

$$\text{and } F \sin a + F' \sin a' + F'' \sin a'' +, \&c = 0$$

SECT.

SECT. III.

COMMUNICATION OF MOTION BY IMPULSE.

68. THE action and reaction of bodies on one another are equal.

This is usually termed the SECOND LAW OF MOTION. When expressed more precisely, it involves two distinct propositions.

1. When motion is communicated by collision or impulse, the quantity of motion gained by the one body in any direction, is just equal to that which is lost by the other in the same direction.

2. When motion is communicated without apparent contact, as in the case of gravitation, and of the phenomena ascribed to attraction or repulsion, the quantity of motion gained by the one body is just equal to that which is gained by the other, but in an opposite direction.

The first of these propositions, involves in it the first law of motion, and when so understood, it properly expresses what is called the INERTIA OF BODY. Like the first law of motion, it follows necessarily from the impenetrability of body.

The second proposition is only known to us by experience.

The demonstration of the first depends on the following theorem.

69. If two bodies meet with velocities that are inversely as their masses, and directly opposed to one another, there will be an equilibrium.

If the bodies are equal, and have equal velocities, the proposition is evident.

If A be double of B, but have only half the velocity of B; then A may be divided into two parts equal to B, each of which will have half the velocity of B. One of these striking against B, would destroy half its velocity; and the other striking against it, would destroy the other half.

The same is true whatever multiple the one body be of the other, and therefore also whatever ratio the one bear to the other.—Vid. D'ALEMBERT, *Dynamique*, § 46. p. 51.

Hence bodies that have equal quantities of motion, have equal forces, or equal powers to produce motion.

70. If A and B be the masses of two bodies that move in the same straight line toward the same parts with the velocities a and b; and if A and B are hard or soft bodies, so that after collision they go on together, and if v be the velocity after collision,

$$v = \frac{Aa + Bb}{A + B}.$$

If b is negative, so that the bodies meet one another, the velocity after collision or $v = \frac{Aa - Bb}{A + B}$; and if $Aa = Bb$, the velocity after collision is nothing, as before demonstrated.

When the bodies are equal, these two formulas become $\frac{a+b}{2}$ and $\frac{a-b}{2}$.

If B is at rest, or $b = 0$, the formula is $v = \frac{Aa}{A+B}$.

If B is an immoveable obstacle, it may be considered as a body that is infinite in respect of A, so that the denominator $A + B$ is infinite, and therefore $v = 0$.

71. If A overtake B, the *velocity* lost by A is $\frac{B(a-b)}{A+B}$; and the *motion* lost by it is $\frac{AB(a-b)}{A+B}$; the *velocity* gained by B is $\frac{A(a-b)}{A+B}$, and the *motion* gained by it is $\frac{AB(a-b)}{A+B}$, the same that was lost by A.

When B meets A, b becomes negative, and so the velocities gained or lost, are $\frac{B(a+b)}{A+B}$, $\frac{A(a+b)}{A+B}$.

72. If

72. If the two bodies before collision move in lines making a given angle with one another, their path after collision may be found, on the principle, that there is no change made in the total quantity of the motion of bodies by their action on one another, in whatever direction their motion be estimated.

If AC (fig. 4.) represent the velocity and direction of the motion of A, BC the velocity and direction of the motion of B, before collision; CG their joint path, and their velocity after collision: Let BD and GF be perpendicular to AC; then, because the quantity of motion in the direction AC remains the same, and also in the direction BD, we have the equations,

$$A.\,AC + B.\,DC = (A+B)\,FC; \text{ and}$$

$$B.\,BD = (A+B)\,FG.$$ Hence CG may be found.

73. When the bodies between which the collision takes place are elastic, that is, when they are such, that, after suffering an alteration in their figure by collision, they resume that figure with a force that makes them separate from one another; the result is considerably different from the preceding. A body is said to be perfectly elastic, when, in restoring its figure, as much motion is produced, as was destroyed in altering it.

74. If A and B are two perfectly elastic bodies, moving with the velocities a and b in the same direction, so that A overtakes B, the velocity of A after the stroke is $\frac{Aa - Ba + 2Bb}{A + B}$; and that of B, $\frac{2Aa - Ab + Bb}{A + B}$, or $\frac{Bb - Ab + 2Aa}{A + B}$.

The investigation turns on this; that the body which overtakes the other must lose twice as much velocity as if it were hard; and the other gain twice as much.

If the bodies move in opposite directions, one of the velocities, as b, must be made negative.

If the bodies are equal, the velocity of A after the stroke is $= b$, and that of B is equal a. The bodies therefore exchange velocities.

When the velocity of either body, after collision, comes out negative, we are to understand its direction to be contrary to that which is accounted affirmative.

75. The difference of the velocities of two perfectly elastic bodies, is the same before and after collision; but the body which had the greatest velocity before collision, has the least after it.

If A and B be the masses of the bodies, a and b their velocities before collision, a' and b' their velocities after it; $a - b = b' - a'$. Hence, also, $a + a' = b + b'$.

Hence, too, the relative motion of the bodies is not changed by their collision.

76. In perfectly elastic bodies, the sum of the products made by multiplying each of the bodies into the square of its velocity, is the same after collision that it was before it.

For as the quantities of motion before and after collision are the same, (68,)

$$Aa + Bb = Aa' + Bb',$$

$$\text{or } A(a - a') = B(b' - b);$$

$$\text{and since, } a + a' = b' + b,$$

$$\text{therefore } A(a^2 - a'^2) = B(b'^2 - b^2),$$

$$\text{or } Aa^2 + Bb^2 = Aa'^2 + Bb'^2.$$

See another demonstration, MACLAURIN's *Account of* NEWTON's *Discoveries*, Book II. chap. iv. § 12.

77. If between two unequal elastic bodies A and C, a third B be interposed; and if the least A, be made to strike with any given velocity on B, the motion communicated to C will be the greatest possible, when B is a mean proportional between A and C.

It is easily shewn, from § 74. that the velocity communicated to C is =

$$\frac{4ABa}{(A+B)(B+C)} = \frac{4Aa}{\left(\frac{A}{B}+1\right)(B+C)} = \frac{4Aa}{A+B+\frac{AC}{B}+C}.$$

This fraction is a maximum, when the denominator $A + B + \frac{AC}{B} + C$ is a minimum, that is, since A and C are given, when $B^2 = AC$, or when B is a mean proportional between A and C.

The velocity of C is $\frac{4Aa}{A + 2\sqrt{AC} + C}$

$$= \frac{4Aa}{(\sqrt{A} + \sqrt{C})^2}, \text{ and the motion of C}$$

$$= \frac{4ACa}{(\sqrt{A} + \sqrt{C})^2}.$$

If the number of the bodies in geometrical progression be increased without limit, the quantity of motion communicated to the last, from a given quantity of motion in the first, however small, may also be increased without limit. Notwithstanding this, as all the bodies move backward after collision but the last, if they form an increasing series, the sum of all the motions in the direction of the first mover continues $= Aa$. Also the sum of the products of each body, into the square of its velocity, after collision, remains as it was before, equal to Aa^2.

78. If an elastic body strike against an immoveable plane, it will be reflected from it, so that its direction before and after collision will make equal angles with the plane, but toward opposite sides.

If the elastic body strike on the plane CD (fig. 5.) in the direction AB, it will be reflected in the direction BE, so that the angles ABC, EBD shall be equal.

79. If two elastic spherical bodies A and B, (fig. 6.) moving in directions, and with velocities represented by the lines AA′ BB′, come into contact at A′, B′, and if A′B′ be the line which joins their centres at that moment: Then if the velocity of A be resolved into two, AC, A′C, the one perpendicular to A′B′, and the other parallel to it; and if the velocity of B be resolved in like manner into BD, B′D; after collision A will retain the velocity represented by AC, and B that represented by BD unchanged; but the velocities AC, BD, will be changed by the collision, into others which may be found by article 74, and if these be *a′* and *b′*; then the path of A after the stroke will be the diagonal of a rectangle under A′E (= AC) and *a′*, and the path of B the diagonal of a rectangle under B′F (= BD) and *b′*.

Thus the paths of the bodies after collision are determined.

80. If the bodies are imperfectly elastic, that is, if, in restoring their figures, they do not generate velocities equal to those which were destroyed in altering them, their motions after collision are not

the same that are here determined. They may be investigated, however, on similar principles.

When the bodies are imperfectly elastic, the sum of the products of bodies into the squares of their velocities does not continue the same after collision that it was before it. The quantity of motion, however, estimated in a given direction, remains the same in every case.

There is another law that extends to the collision of all bodies, whether elastic, unelastic, or imperfectly elastic. It is, that if each body be multiplied into the square of the change that has taken place in its velocity by the collision, the sum of these products is a *minimum*; that is, using the preceding notation $A(a-a')^2+B(b-b')^2$ is the least possible, or is less in the case which actually takes place in nature, than if the differences between a and a', b and b', were other than they are. This law was discovered by MAUPERTUIS. It is not demonstrated from principle, but is collected from the preceding propositions by induction.

SECT. IV.

OF MOTION EQUABLY ACCELERATED OR RETARDED.

81. WHEN the motion of a body varies either in direction or velocity, as this cannot arise from any thing in the body itself, it must be ascribed to the action of an external cause or force.

The simplest case is, when the direction remains the same, and when the velocity only varies.

82. When the velocity of a body changes, the cause of that change is called an accelerating or retarding force, and is measured by the change of the velocity produced in an indefinitely small portion of time, or by the increment of the velocity divided by the increment of the time.

If F be the accelerating force, $\dot{v}$ the momentary increment of the velocity, $\dot{t}$ that of the time,

$$F = \frac{\dot{v}}{\dot{t}}.$$

Hence also $\dot{v} = F\dot{t}$, and $\dot{t} = \frac{\dot{v}}{F}$.

It has been disputed, whether this expression of the force be a necessary truth, or one known only from experience. D'ALEMBERT, *Elemens de Phil. Melanges*, tom. 4. p. 197. It seems, however, to be in fact a definition, or an assumption, and not a theorem. We have no distinct idea attached to the word FORCE, which we can compare with that which is conveyed by the formula $\frac{\dot{v}}{\dot{t}}$, in order to see whether there is a necessary agreement between them or not. Therefore that $F = \frac{\dot{v}}{\dot{t}}$ is not a theorem. But as the quantity $\frac{\dot{v}}{\dot{t}}$ is of great importance, and frequent recurrence in mechanical investigations, it is convenient to have a term to denote it. Though any term might be employed for this purpose; yet as the thing called FORCE, is conceived to be always greater, the greater the change of velocity which it produces in a given time, or to increase and diminish with $\frac{\dot{v}}{\dot{t}}$, we may, without diverting the word FORCE from its usual signification, employ it to denote the quantity $\frac{\dot{v}}{\dot{t}}$, or the momentary increment of the velocity divided by the corresponding increment of the time. The word Force has in reality, in dynamics, no other signification than this; the one may be everywhere substituted for the other, and an entire treatise of dynamics might be written, in which the word Force would not once occur.

Retarding Forces are here included under the head of Accelerating Forces, as the same force may accelerate or retard, according as the velocity of the moving body is in one direction or in the opposite.

83. If a body be continually urged by the same accelerating force, in the same direction, its motion will be uniformly accelerated, or its velocity will increase proportionally to the time.

The momentary increment of the velocity will be always the same, from the definition of accelerating force; and as none of the velocity ever acquired can be lost, by the first law of motion, therefore, it increases continually at the same rate.

This is applicable particularly to the case of bodies falling by their gravity to the earth. The descent of these bodies, their velocity being found to increase uniformly, affords a direct and experimental proof of the first law of motion.

The propositions that follow are all immediately applied to falling bodies; but they are equally applicable to all rectilineal motions produced by the constant action of the same force.

84. If the velocity which a heavy body falling from rest, acquires in the first second of its motion be called g, then after any time t, v its velocity will be $g \times t$; and if s be the entire space through which it has descended, $2gs = v^2$.

Since $v = gt$, and therefore $v^2 = g^2 t^2$,

$2gs = g^2 t^2$, or $s = \frac{1}{2} g t^2$.

Hence if $t = 1''$, and if s' be the space fallen through in one second, $g = 2s'$; or the velocity is equal to twice the space fallen through in one second.

The space through which a body falls in the first second of time is sixteen feet and one inch nearly, and therefore $g = 32.\frac{1}{6}$ nearly.

85. If the velocity acquired by a body falling from rest, were at any point to become uniform, it would carry the body, in an equal time, over twice the distance which it has actually fallen through.

86. If a body be projected directly upwards with any velocity c, its motion will be uniformly retarded, or it will lose in every second a velocity equal to g, the velocity generated by gravity in a second.

After the time t, the velocity of the body is $c - tg$; and the time at which the body reaches its greatest height where $v = 0$, is $\frac{c}{g}$.

87. The greatest height to which a heavy body can ascend, when projected directly upwards with the velocity c, is $\frac{c^2}{2g}$, the same height from which it must have fallen to acquire the velocity c; and in all points, both of its ascent and descent, the relation between the time and the height h is expressed by the equation

$$(c-gt)^2 = \left(\frac{c^2}{2g} - h\right) 2g, \text{ or}$$

$$c - gt = \sqrt{c^2 - 2gh}.$$

This equation gives $t = \frac{c}{g} \pm \frac{1}{g}\sqrt{c^2 - 2gh}$ The reason why two different values of t correspond to the same value of h is, that the body is at the same height at two different times, one in its ascent, and another in its descent.

The sum of the two values of t is $\frac{2c}{g}$, or twice the time of the ascent.

The difference between them is $\frac{2}{g}\sqrt{c^2 - 2gh}$.

88. If two bodies A and B, of which A is the heavier, be suspended over a pulley moveable

able about a fixed centre, A will descend, and will be accelerated by a force proportional to $\frac{A - B}{A + B}$, or to the difference of the weights divided by their sum.

If the natural force of gravity be expressed by g, the velocity which it generates in a second; then the acceleration of the bodies will be $\frac{A - B}{A + B} \times g$.

We may therefore diminish the intensity of gravitation at pleasure, or produce an accelerating force less than the force of gravity in any assigned ratio.

It is on this principle that Mr Atwood's machine, for reducing the theory of accelerated motion to the test of experience, is contrived.

The acceleration of falling bodies was discovered by Galileo, *Dialogo* III. *de Motu locali*, *Opere*, tom. 3. p. 98. edit. Padova, 1744.

Sect.

SECT. V.

MOTION OF PROJECTILES.

89. IF a heavy body be projected in the direction of a straight line, not perpendicular to the horizon, it will describe a parabola situate in the vertical plane passing through that straight line, and having its axis perpendicular to the horizon.

> The straight line in the direction of which the body is projected, touches the parabola in the point of projection.
>
> Gravity is here supposed to have its intensity and its direction the same at all points, within the range of the same projectile. We also abstract from the resistance of the air.
>
> The perpendicular height of the directrix of the parabola above the point of projection is $\frac{c^2}{2g}$.
>
> This proposition was discovered by GALILEO, and demonstrated, *Dialogo* IV. *de Motu Projectorum.*

90. If from the point of projection there be taken upwards, in a vertical line, a part equal to $\frac{c^2}{2g}$, (*c* being the velocity with which the body is projected,)

projected,) the directrix of the parabola described by the projectile will pass through that point; and the velocity of the projectile in every point of its path, will be the same that it would have acquired by falling freely from the height of the directrix.

All the parabolas described by bodies projected from the same point, and with the same velocities, though with different elevations, have the same directrix; and have their foci in the circumference of the same circle.

91. If a body be projected with the velocity c, and in a direction that makes an angle E with the horizon, the distance at which it will strike the horizontal plane, (or what is called its *horizontal range*), will be equal to twice the perpendicular height to which it would be carried by its initial velocity, multiplied into the sine of double the angle E; or $= \frac{c^2}{g} . \sin 2 E$.

Hence the greatest horizontal range is, when the direction of the projectile makes an angle of 45° with the horizon: Also with respect to every other range, there are two angles which will give the same range, the one as much less than 45° as the other is greater.

92. The greatest altitude above the point of projection, to which a projectile can ascend, is equal to the height corresponding to its initial velocity, multiplied into the square of the sine of the elevation, that is, to $\frac{c^2}{2g} \sin E^2$. Also the time of its flight from the moment of projection till it strike the horizontal plane passing through the point of projection, is $\frac{2c}{g} \times \sin E$.

> **The greatest height to which the projectile ascends is therefore as the square of the velocity, multiplied into the square of the sine of the elevation; and the time of the flight is simply as the velocity, multiplied into the sine of the elevation.**

93. The greatest distance to which a projectile can go in the direction of any plane whatever, passing through the point of projection, is when its direction bisects the angle which the given plane makes with the vertical.

94. The points of greatest distance at which a projectile, thrown with a given initial velocity, will strike the different planes in the last proposition, are all in a parabola given in position.

This parabola has its axis perpendicular to the horizon, and touches all the parabolas described by the projectile, when thrown in the same vertical plane, and with the same velocity, but at different elevations.

Sect. VI.

OF MOTION ACCELERATED OR RETARDED BY A VARIABLE FORCE.

95. The changes of motion, whether in velocity or direction, are always made gradually, and never, as it is expressed, *per saltum*. Though they may not, like those treated of in the last section, be exactly proportional to the time, they are always proportional to a *function* of the force and the time jointly, which *function* vanishes when either the time or the force is equal to nothing. This follows from the fundamental equation of Dynamics, $F = \frac{\dot{v}}{\dot{t}}$ or $\dot{v} = F\dot{t}$.

The above is called the LAW OF CONTINUITY, which, in what respects free motions, is never violated. This conclusion agrees perfectly with all the instances in which accelerated or retarded motion can be traced by the senses, and therefore we might conclude, even from analogy, that it holds in those cases where the progress of acceleration escapes observation. When a ball, from being at rest, issues almost in an instant from the mouth of a cannon, with the velocity of 1800 feet in a second, it is not to be imagined, that it has acquired the whole of that velocity, or any part of it, suddenly, and without the lapse of time. Could we divide time into small enough portions, and did we perfectly understand the Law of the Force produced by the inflammation of gun-powder, we should be able to determine the point of time, and the place in the interior of the gun, at which the ball had any assignable velocity from 0 to 1800 feet *per* second; when, for instance, it had a velocity of one foot *per* second; of two, of 100, of 1200, &c. The same holds of the direction of motion; a body from moving in one direction, does not come to move in another, without describing a portion of a curve, and taking in its motion every possible direction from the one to the other.

The notion of the continuity of motion seems first to have occurred to GALILEO. LEIBNITZ introduced it as a leading principle in his philosophy, and he proved the necessity of it by a metaphysical, but conclusive argument. If a body receive an increase of its motion without the lapse of time, then the same body, at the same indivisible instant, is in two different states, which is impossible. If, for instance, it is at the beginning of the motion that the *saltus* is made, the body is at the same instant both at rest and in motion.

96. Let AB (fig. 7.) be a straight line, along which a body is accelerated by forces directed from A to B; let the perpendicular AD be equal to the velocity which a body, accelerated by the force at A, would acquire in one second; and let DEF be a curve so related to AB, that at any point whatever C, the ordinate CE may be to AD as the accelerating force at C to the accelerating force at A; then, if the body begin to descend from rest at A, the square of its velocity at any point C, is double of the curvilineal area ACED.

For, by the construction, CE is proportional to the force at C $= \frac{\dot{v}}{\dot{t}}$, v being the velocity which the moving body has acquired at C, and t the time of the descent from A to C. Now C c is the momentary increment of AC the space, and is therefore $= v\,\dot{t}$; therefore, CE × C $c = v\,\dot{v}$, and $2\,\mathrm{CE} \times \mathrm{C}\,c = 2\,v\,\dot{v}$. But CE × C c is the momentary increment of the area ACED, and $2\,v\,\dot{v}$ is the momentary increment of v^2; therefore the square of the velocity of the moving body, and twice the area of the curve ACED, increase at the same rate, and begin to exist at the same time; therefore they are equal. NEWTON's *Princip. Lib.* I. *Prop.* 39.

If the body begin to move from A with a certain velocity, then the difference between the square of its velocity at C and at A is equal to twice the area ACED.

In like manner, if the body be projected upward with a given velocity, the difference between the square of the initial velocity, and the square of its velocity at any point C, will be double of the area BCEF; and if the motion of the body be destroyed when it has ascended to a certain point A, the square of the initial velocity will be double of the area BADF.

97. If a body be accelerated or retarded by any number of forces acting in succession, and each continuing its action while the body moves over a certain distance; the sum that is made up by multiplying each of these forces into the distance over which it acts, and adding those products together, will be proportional to the difference of the squares of the velocities of the body at the beginning and end of the action of those forces.

98. Hence if the body is accelerated from rest, the sum collected, as in the last proposition, will be as the square of the velocity acquired; and if it be retarded or resisted till it come to rest, the sum will be as the square of the initial velocity.

Thus if F, F′, F″, F‴, &c. are the forces that act in succession, s, s', s'', s''', the distances or spaces over which they act; v the velocity at the beginning of the action, v''' at the end of it,

$$F s + F' s' + F'' s'' + F''' s''' \doteqdot + v^2 \mp v'''^2.$$

And if $v''' = 0$, then

$$F s + F' s' + F'' s'' + F''' s''' \doteqdot v^2.$$

99. If the force which accelerates or retards in the last proposition, be uniform, the *distance* to which the body will go before it acquire or lose a given velocity, will be as the square of that velocity.

On this is founded that estimation of the *force* of moving bodies, which is known by the name of the VIS VIVA, and which makes that force proportional to the square of the velocity. The truth is, that the effect of a body in motion may be measured either by the *distance* it goes to, or by the *time* that elapses before a resistance of uniform intensity reduce it to rest. If the effect be measured in the first of these ways, it will be found to be as the square of the velocity; if in the second, as the velocity simply. Both these measures may be considered as correct, and are not inconsistent when rightly understood. The former makes the effect F s, the latter F t, retaining the significations already given to these letters. The same term, however, ought not to be indiscriminately applied to two things so essentially different. It is most conformable to the use made of the word Force, in the fundamental investigations of Dynamics, to take the quantity F t for the measure of it, and therefore to suppose it proportional to the velocity simply. This is the only way in which the language of the science can be rendered perfectly consistent and free from ambiguity. But as the distance to which a body goes before a given resistance destroys its motion, is often an object of inquiry, it may be proper to consider that distance as a measure of a certain power residing in the moving body. It has been proposed to distinguish this power by the term *Impetus*.

This part of Dynamics would lead to the consideration of Central Forces, but as this is a subject of some difficulty, and is most interesting when connected with astronomy, we shall defer treating of it till we enter on the explanation of the Planetary Motions.

MECHANICS.

MECHANICS.

Sect. I.

OF THE CENTRE OF GRAVITY.

100. If a body fixed to a plane be preserved *in equilibrio* by a force having a given direction, and attached also to a given point in the plane, it will be kept *in equilibrio* by the same force, when attached to any other point whatever in the line of its first direction.

Let the force be attached by a thread to a given point in the plane; while the tension of the thread and its direction continue, the equilibrium must continue. Suppose, then, that any other point in the thread is fixed down to the plane, it is evident, that the tension of the remaining part will not be changed by that circumstance; and therefore the equilibrium must remain.

Carnot has given a different proof of this proposition. *Principes de l'Equilibre*, &c. § 77.

101. If two heavy bodies be fixed to a plane moveable about a given point, the bodies will be *in equilibrio* if that point divide the distance between them in the inverse ratio of their weights or quantities of matter.

A demonstration of this proposition may be derived from the composition of motion, as is done by NEWTON, *Principia Mathematica*, p. 15. and by HAMILTON, *Philosophical Essays*, p. 145.

A different demonstration was given by ARCHIMEDES, and has been improved by MACLAURIN in his *Account of Newton's Discoveries*, and by VINCE in the *Philosophical Transactions*. See also WOOD's *Mechanics*, Art. 73.

102. The effect of any heavy body to produce an angular motion in any system of bodies about a given centre, is proportional to the product of the mass of the body into the perpendicular drawn from the centre to a vertical line passing through the body.

The axis of motion of any body is a straight line, which remains fixed while the body is free to revolve round it.

103. The product of any heavy body into its distance, from a vertical plane passing through the axis of motion, is called the *momentum of the body* relatively to that axis; and in general, if, through the axis of motion, a plane be made to

pass parallel to the direction of any force, the product of that force into its distance from the same plane is called the *momentum of the force.*

104. Any system of bodies being given, a point may be found, such, that if any plane pass through it, the sum of the momenta of the bodies on one side of the plane, relatively to the point found, shall be equal to the sum of the momenta of the bodies on the other side of the plane, relatively to the same point.

> The point thus found, is called the *centre of gravity* of the system.
>
> Every system of bodies, therefore, and every individual body, (as a body may be regarded as a system of particles or infinitely small bodies), has a centre of gravity; and that centre remains the same while the bodies retain their position relatively to one another, however the whole system may change its place relatively to the horizon.

105. In any body or system of bodies, if the centre of gravity be sustained, the whole will remain at rest.

> In other words, the body or system of bodies has no tendency to angular motion round its centre of gravity.

106. The distance of the centre of gravity of any system of bodies from a given plane, is equal to

the sum of the products of all the masses into their distances from the plane, divided by the sum of the masses.

As a single body may be regarded as a system of particles, this proposition applies to such bodies, and is therefore universal.

If the bodies be A, B, C, D, &c. and their perpendicular distance from a given plane a, b, c, d, &c. x the distance of the centre of gravity from the same plane, $x = \frac{Aa + Bb + Cc + Dd}{A + B + C + D}$.

107. The effect of any number of bodies to produce motion about a given point, is the same as if those bodies were united in their common centre of gravity.

108. When a heavy body is at liberty to move about a fixed point, it will not be at rest till its centre of gravity is either the highest or the lowest possible.

A body cannot rest on any base, unless a perpendicular to the horizon from its centre of gravity fall within the base.

The centre of gravity of plain figures, and of solids, may be found by the foregoing propositions. A few of the simplest and most elementary of these problems may serve as examples.

109. To find the centre of gravity of any number of bodies placed in the same straight line at given distances from one another.

The centre of gravity is found by Art. 106.

If the bodies are A, B, C, D, and their distances from a given point in the straight line in which they are supposed to be placed, a, b, c, d, the distance of their centre of gravity from the same point, is $\frac{Aa + Bb + Cc + Dd}{A + B + C + D}$.

If the bodies are on different sides of the point to which their distances are referred, those on one side being accounted *affirmative*, those on the opposite must be reckoned *negative*.

110. Any number of bodies being given in position, to find their centre of gravity.

The bodies must be referred to three planes given in position, cutting one another at right angles, one of them horizontal, and, of course, the other two vertical.

Let the bodies be A, B, C, and D, their distances from the given horizontal plane a, b, c, d; their distances from one of the vertical planes a', b', c', d'; and from the other a'', b'', c'', d''; then if we take $x = \frac{Aa + Bb + Cc + Dd}{A + B + C + D}$, the centre of gravity of

of the system is in a horizontal plane at the distance x from the given horizontal plane.

Take also, $x' = \frac{A a' + B b' + C c' + D d'}{A + B + C + D}$, and the centre of gravity is in a plane parallel to the first of the two vertical planes, and distant from it by the line x'.

Lastly, take, in the intersection of these planes, a point distant from the second vertical plane by a quantity $x'' = \frac{A a'' + B b'' + C c'' + D d''}{A + B + C + D}$.

This point will be the centre of gravity of the given bodies, as is evident from § 106.

111. To find the centre of gravity of a triangular plane, all the points of which are supposed to gravitate equally.

From two of the angles of the triangle, draw lines bisecting the opposite sides. Their intersection is the centre of gravity.

Each of these lines is divided by the point of intersection, in the ratio of 2 to 1.

112. To find the centre of gravity of a given pyramid.

Draw a straight line from the vertex to the centre of gravity of the base, and divide it in the ratio of three to one, the greatest segment being next the vertex. The point so found is the centre of gravity of the Pyramid.

This construction applies to a pyramid of any number of sides, and therefore also to a Cone.

113. To find the centre of gravity of any plane mechanically.

Suspend it by a given point in or near its perimeter, and when it is at rest, draw across it a vertical line passing through that point. Suspend it in like manner by another point, and draw a vertical line as before. The intersection of these lines is the centre of gravity of the plane.

If the body is of three dimensions, the same process may be followed; but three suspensions will be necessary.

This construction is referred to by PAPPUS ALEXANDRINUS, *Collect. Math.* Lib. VIII. Prop. i.

Of the Motion of the Centre of Gravity.

In the preceding propositions, the centres of gravity of bodies or systems, in which the parts preserved the same relative position, have only been considered. It is evident, however, that in any system, though the parts be not fixed in the same relative position, the centre of gravity for any given position may be found; and there are some general properties of the centres so found, that are very important in mechanics.

114. If any number of bodies move uniformly in straight lines, their common centre of gravity will

also move uniformly in a straight line, or will remain at rest.

Newtoni *Principia*, *Lib.* 1. *Lemma* 23.

115. The mutual action of bodies does not change the state of their centre of gravity, either as remaining at rest, or as moving uniformly in a straight line.

116. The quantity of motion in any system of bodies estimated in a given direction, remains constantly the same, whatever may be the mutual action of these bodies on one another.

> Both this and the preceding proposition are consequences of the equality that takes place between the action and re-action of bodies. The quantity of motion in any system is the same as if all the parts of it were united in the centre of gravity of the whole.

117. If there be a system of bodies acting any how on one another, and if at any point of time we compute the motions which these bodies would have in the succeeding instant, were they all free from their mutual action; and if we also compute the motions, which, in consequence of their mutual action, they really have in that instant, the motions which must be compounded with the first of these, in order to produce the second, are such as, if they acted on the system alone, would pro-

duce no motion at all, or would be *in equilibrio* with one another.

The principle here laid down was discovered by D'Alembert. It is a result of the equality of action and re-action, or rather, it is that very equality expressed with perfect generality and precision. It was supposed before, that the motions combined with the first of those above described, in order to produce the second, must, when added together, be equal to nothing; whereas it is not the motions simply, but their momenta, which, when added together, must be equal to nothing.

This principle is of great importance for deducing the laws of constrained from those of free motion, and affords the only general method of determining the manner wherein motion distributes itself through all the parts of a system to which it is any how communicated. It may very properly be called the *Principle of the Distribution of Motion.*

A simple instance of its application is afforded by the following problem.

118. Let two bodies connected by a straight rod, without weight, be moveable about an axis passing through their centre of gravity; and let a motion be communicated to one of them, which, were it detached and single, would give it a velocity c; required, the angular velocity with which the bodies will begin to revolve?

Sect.

SECT. II.

OF THE MECHANICAL POWERS.

119. A MECHANICAL power, is an instrument by which the effect of a given force is increased, while the force itself remains the same.

The simple mechanical powers into which more complex machines are resolved, are these: 1. The Lever; 2. The Wheel and Axle; 3. The Pulley; 4. The Wedge; 5. The Screw; 6. The Funicular Machine.

Of the Lever.

120. DEF.—A lever is an inflexible rod, moveable about a centre or fulcrum, and having forces applied to two or more points in it.

The simplest state of the lever is, when there are only two forces; and of this there are two cases: in the first, the fulcrum is between the points where the forces are applied, and the forces are directed the same way; in the second, the points to which the forces are applied, are on the same side of the fulcrum, and the forces are directed opposite ways.

In treating of the lever, it is usual to distinguish the forces by the names of the Power and the Weight, or the Power and the Resistance,—terms that have a reference to the intention with which the machine is used, not to any real difference in the action of the forces.

We shall begin with abstracting entirely from the weight of the lever itself.

121. If two forces be applied to a lever having equal and opposite *momenta*, that is, tending to produce motion in opposite directions, and being inversely as the perpendiculars drawn to their direction from the fulcrum, they will be in equilibrium with one another.

When the forces are parallel, this proposition coincides with the property of the centre of gravity, § 101. When they are oblique, the case is reduced to the preceding by the resolution of forces.

122. If any number of forces be applied to a lever, there will be an equilibrium, if the sums of the opposite momenta be equal to one another.

This proposition also coincides with a property of the centre of gravity (§ 105).

123. When any number of forces are perpendicularly applied to a lever, the fulcrum is loaded

with the sum of the forces, if they are all in the same direction, and with their difference, if they are in opposite directions.

124. When there is an equilibrium in the lever, if each force be multiplied into the velocity which the point of its application would have, if motion were to take place, the sum of all the products so formed is equal to nothing.

Forces that act in opposite directions are here accounted positive and negative. This is always to be understood when the sums of forces or of momenta are said to be equal to nothing.

This property of the lever is common to all the mechanical powers, and, indeed, to all machines whatsoever. It is known by the name of the principle of *Virtual Velocities*, and consists in this, that if the equilibrium of a machine be disturbed, by a quantity indefinitely small, and if the velocity of each force be multiplied into its quantity, the sum of these products, reckoning the forces which are in opposite directions, positive and negative with respect to one another, will be equal to nothing. This principle was suggested by BERNOUILLI, and is of great use in mechanical investigations. It admits of a general, though indirect, demonstration.

The velocities compared, being not *actual*, but such as *would* take place if a certain event, the subversion of the equilibrium, were to happen, are properly denominated *virtual*.

The different cases of the lever, whether it be straight or crooked, and the forces perpendicular or oblique, are all comprehended in the preceding propositions. The case of oblique forces admits of a very concise expression in terms of the angles.

125. If two oblique forces be applied to the extremities of a straight lever, there will be an equilibrium between them, if they be inversely as the lengths of the arms by which they act, multiplied into the sines of the angles which their directions make with the lever.

It is evident that this proposition is general, and comprehends the case of perpendicular forces.

126. If a beam of any form be kept in equilibrium by three forces applied to different points in it, these forces must either be parallel, or converge to the same point.

If at the point to which the three lines converge, we take parts in those lines proportional to the forces, they will form the sides and the diagonal of a parallelogram; and if at the point in which this diagonal intersects the beam, we suppose a fulcrum to be placed; the beam will be converted into an ordinary

nary lever, and the weight sustained by the fulcrum will be to either of the other two forces, as the diagonal of the parallelogram to the side which corresponds to that force: also the two extreme forces will be to one another inversely as the sines of the angles which their directions make with the above diagonal.

127. If we would allow for the weight of the lever itself, we must suppose its weight to be united in its centre of gravity, and to act there as a third force, added to the power or the resistance, according to the side of the fulcrum on which it is placed.

128. When a beam carrying a weight, is supported in a horizontal position by two props, the weights which the props sustain are inversely proportional to their distances from the centre of gravity of the weight.

129. If a weight W, be sustained on a horizontal plane by three props, (not in a straight line), the pressure on each will be the same as if a single weight were laid on it, so that the sum of all the three weights were equal to W, and their common centre of gravity the same with the centre of gravity of that body.

If the props be A, B, C, (fig. 8.), and if W be placed with its centre of gravity at D, and if ADG, BDF, CDE, be drawn, the pressure

$$\text{on } A \text{ is } \frac{DG}{AG} \times W,$$

$$\text{on } B \text{ is } \frac{DF}{BF} \times W,$$

$$\text{on } C \text{ is } \frac{DE}{CE} \times W.$$

When D coincides with the centre of gravity of the triangle ABC, the pressure on each of the props is the same.

If W be supported by more than three props, the problem appears to be indeterminate, or to admit of innumerable solutions.

Nevertheless, if the centre of gravity of W, as it rests on the plane, be the same with the centre of gravity of the figure made by joining the tops of the props by straight lines, the pressures on the props are all equal to one another.

The lever is the simplest of the mechanical powers, and appears to be the first that was attempted to be explained. ARISTOTLE has treated of it in his Mechanical Questions, and has even endeavoured to reduce some other of the mechanical powers to the lever. The first accurate explanation, however, was given by ARCHIMEDES, *Lib.* I. *de Æquiponderantibus*, *Prop.* 6.

The lever is used in mechanics in various forms and combinations. The compound lever is, where one

lever is made to turn another; and then an equilibrium takes place, when the weight is to the power as the product of all the arms, taken alternately, beginning with that to which the power is applied, to the product of all the other arms.

The common apparatus for raising a draw-bridge, is a combination of a lever of the first kind, with a lever of the second; the two levers are parallel, and remain *in equilibrio* in all situations.

In the construction of the animal body, this mechanical power is much employed. The bone is the lever, the muscle is the power, or rather the medium through which the power acts, and the joint is the fulcrum. The insertion of the muscle, or the point to which the power is applied, is usually nearer the joint or fulcrum than the weight to be raised is. A different structure might have produced more strength, but would have diminished the activity of the animal, and the velocity of its motion.

Of the Balance.

130. The balance is a lever with equal arms, used for comparing weights with one another, and, when well constructed, must have the following properties: 1. It should rest in a horizontal position, when loaded with equal weights. 2. It should have great sensibility, that is, the addition of a small weight in either scale should disturb the equilibrium, and make the beam incline sensibly

from the horizontal position. 3. It should have great stability, that is, when disturbed, it should quickly return to a state of rest.

That the first requisite may be obtained, the beam must have equal arms; and the centre of suspension must be higher than the centre of gravity. Were these centres to coincide, the beam, when the weights were equal, would rest in any position, and the addition of the smallest weight would overset the balance, and place the beam in a vertical situation, from which it would have no tendency to return. The sensibility in this case, would be the greatest possible; but the other two requisites of level and stability would be entirely lost. The case would be even worse, if the centre of gravity were lower than the centre of suspension, as the balance, when deranged, would make a revolution of no less than a semicircle. When the centre of suspension is higher than the centre of gravity, if the weights be equal, the beam will be horizontal, and if they be unequal, it will take an oblique position, and will raise the centre of gravity of the whole, making the momentum on the side of the lighter weight equal to that on the side of the heavier, so that an equilibrium will again take place.

The second requisite is the sensibility of the balance, or the smallness of the weight by which a given angle of inclination is produced. If a be the length of the arm of the balance, and b the distance between the centre of suspension and the centre of gravity, P the load in either scale, and W the weight of the beam, the sensibility of the balance is as $\frac{a}{b(2P+W)}$; it is therefore greater, the greater the length of the arm, the less the distance between the two centres, and the less the weight with which the balance is loaded.

Lastly, The stability, or the force with which the state of equilibrium is recovered, is proportional to $(2P+W)b$, the denominator of the preceding fraction.

The diminution of b, therefore, while it increases the sensibility, lessens the stability of the balance. The lengthening of a will, however, increase the former of these quantities, without diminishing the latter. The above formulas are of great practical utility, because, by means of them, one balance may be made having exactly the same sensibility and stability with another: it is only required, that the ratio of the lengths of the arms should be the same with that which is compounded of the ratios of the distances of the centres of gravity and suspension, and of the weights of the beams.

A balance made by Ramsden for the Royal Society, is capable of weighing ten pounds, and turns with half a grain, the ten millionth part of the weight. Young's *Lectures*, vol. I. p. 125. The descriptions of balances, given by mechanical writers, are generally defective, as they do not give the values of a, b, and W, the quantities on which the merit of the balance depends, and by the knowledge of which similar instruments might be constructed.

In some of the nicest balances, *b* is made variable by means of a small moveable weight.

Consult La Hire, *Traité de Mechanique*, prop. 33. Muschenbroek, § 383, &c. Euler, *Comment. Petrop.* tom. x. p. 1. &c. Magellan, *Journal de Phys.* tom. xvii. (1781), p. 43.

Of the Wheel and Axle.

131. The wheel and axle, consists of a wheel having a cylindric axis passing through its centre, and moveable in two grooves. The power is applied to the circumference of the wheel, and the weight to the circumference of the axle. An equilibrium in this instrument takes place, when the radius of the wheel, multiplied into the power, is equal to the radius of the axle multiplied into the weight; or when the power and weight are inversely as the radii of the circles to which they are applied.

It is supposed here, that the power and the weight both act at right angles to the radius.

The wheel and axle is nothing else but a lever, so contrived as to have a continued motion about its fulcrum. The principle of the virtual velocities is here obviously applicable. If the power act not at the circumference of the wheel, but at the extremity of a handspike inserted in the wheel, the distance

tance of that extremity from the centre of the axis, is to be accounted the radius of the wheel.

The Capstan, the Windlass, and other contrivances of a similar nature, are nothing else than the wheel and axle adapted to particular circumstances.

The mechanical contrivance called a Crank, is a species of wheel and axle. If the force which acts upon a crank presses it directly down and up alternately, the effect, compared with what would take place if the force acted at right angles to the arm of the crank all round, is as twice the diameter of a circle to its circumference, or as seven to eleven nearly.

The combinations of wheels, so useful in mechanics, are generally reducible to the wheel and axle, though the wheel which turns the other is not always upon the same axis with it. The motion, in such cases, is communicated from the one wheel to the other, either by belts or straps passing over the circumference of both; or by teeth cut in the circumference of each, and working in one another.

132. When one wheel moves another in either of these ways, the velocities of their circumferences are equal; and therefore their angular velocities, or the number of revolutions which they make in the same time, are inversely as their radii.

In combinations of wheels communicating motion to one another, it is usual to call the smaller wheel acted on by a larger one, a *pinion;* and the teeth the *leaves* of it. Sometimes the smaller wheel is a *cylinder*, in which the top and bottom are formed by circles, connected by staves inserted in them at equal distances along their circumferences; and it is then called a *lantern*, the staves serving for teeth. It is usual also to employ the wheels and pinions, as in figure 9, where A is a large wheel, driving a pinion *b* on the same axle with the great wheel B; B acts on the pinion *c*, which is on the same axle with the large wheel C; and C drives the pinion *d*, &c.

133. When motion is communicated through a number of wheels and pinions, the angular velocity of the first wheel, is to the angular velocity of the last pinion, as the product made by multiplying together the number of the teeth in every pinion, to the product made by multiplying together the number of the teeth in every wheel; and that same ratio, compounded with the ratio of the radius of the last pinion, to the radius of the first wheel, will give the ratio of the power to the resistance when an equilibrium takes place.

The truth of the first part of this proposition is evident from § 132.; and the truth of the latter part follows from the principle of the virtual velocities.

134. When wheels act by teeth working in one another, notwithstanding that the length and position of the levers by which they act, are continually changing, the force of the one upon the other will remain constant, if the line which is drawn perpendicular to the surfaces of both teeth at the point of contact, pass continually through the same point of the line which joins the centres of the wheels.

It is here necessary to distinguish between what is called the *total*, and the *primitive* radius of a wheel. The total radius extends from the centre to the extremities of the teeth; and the primitive extends only to the point where the teeth would touch, if they were to act merely by contact. If the distance between the centres of the wheels be divided by the point O, (fig. 10.), in the ratio of the number of teeth which the wheels are required to have, AO and BO will be the primitive radii of the wheels, and O is the point through which the perpendiculars in the above proposition must be supposed to pass.

135. There are many different curves, according to which the teeth of wheels might be formed, so as to answer the condition in the last proposition; but that which seems most convenient, and which has been most generally adopted, is the *epicycloid.*

Vid. EULER, *Nov. Com. Petrop.* v. p. 299. and XI. p. 207. One of the curves there proposed, is

the *evolute* of the circle. The same is mentioned, *Encyc. Brit.*

If the circumference of one circle be made to roll along the circumference of another, the curve described by any given point in its circumference, is an epicycloid. The circle which rolls is called the *generating circle*, and that on which it rolls the *base of the epicycloid.* As the former may be supposed to roll either on the outside or the inside of the latter, there are two kinds of epicycloids, distinguished by the names of Exterior and Interior.

136. Let it be required, having given the magnitude of a wheel and pinion, and the numbers of teeth in each, to determine the figure of those teeth. Let CB (fig. 10.) be the primitive radius of the wheel, and BD the base of the tooth ; bisect BD in E, draw CE, and produce it indefinitely ; and with a generating circle, of which the radius is half the primitive radius of the pinion, describe an arch of an epicycloid on the primitive circumference of the wheel as a base, beginning at the point B, and intersecting CE in F. Then BF is the figure of one face of the tooth, and in the same manner may the other FL be determined. When one tooth is formed, it may serve as a model for the rest.

The teeth of the pinion are to be determined in the same manner ; the radius of the generating circle being made equal to half the primitive radius of the wheel.

See CAMUS *on the Teeth of Wheels*, Article 548. p. 53. *et seq*. of the English translation.

Toothed wheels are frequently employed for changing the direction of motion: the directions of their axes are then inclined to one another, and the parts of their rims where the teeth are placed, are cut into frustums of cones, each of which has its vertex in the intersection of the axes of the two wheels. Wheels of this kind are called *bevelled*, or *crown wheels*.

137. The figures of the teeth of bevelled wheels, are determined by an epicycloidal surface, generated by a straight line, on the surface of a cone, while the cone rolls on the conical part of the wheel, its vertex being in the point where the axes of the wheels intersect. The line on the surface of the revolving cone passes through its vertex.

Consult CAMUS *on the Teeth of Wheels*, Article 557. &c. The curve generated by a point on the surface of a revolving cone, is there called a Spherical Epicycloid, as being on the surface of a sphere. Dr BREWSTER, in the *Appendix to Ferguson's Mechanics*, vol. II. p. 232. treats of the construction of bevelled wheels.

Toothed wheels are employed in mechanics, not only for increasing or diminishing velocity, so as to adapt a given power to the purpose of producing a certain effect, as in a common mill, where the slow motion of the water-wheel is made to produce the rapid motion of the millstone, but they are also employed for the purpose of producing angular motions, that shall obtain, with great precision, given ratios to one another. This happens in clock-work, and it is there of importance to determine with accuracy the number of wheels, and the number of teeth in each. Suppose it is required to make one wheel turn exactly 240 times, while another turns once: The smaller wheel, which is to turn the fastest, cannot have fewer than 6 leaves or teeth: The other, therefore, ought to have 6 times 240, or 1440, which is more than can conveniently be given to one wheel. Suppose, then, that three wheels ABC, and three pinions $b\,c\,d$, as in figure 9, are to be employed: Then the angular velocity of A is to the angular velocity of b, as

$$b \times c \times d \text{ to } A \times B \times C,$$

these letters denoting the number of teeth in their respective wheels, therefore

$$A \times B \times C : b \times c \times d :: 240 : 1.$$

Let the number of teeth in the pinions be 7, 5, and 11, then

$$A \times B \times C : 7 \times 5 \times 11 :: 240 : 1;$$

and $A \times B \times C = 7 \times 5 \times 11 \times 240$, or resolving 240 into its simple factors,

$$A \times B \times C = 7 \times 5 \times 11 \times 3 \times 2 \times 2 \times 2 \times 2 \times 5.$$

These factors, combined into three different groups in any order, will give three products that may be substituted for A, B, and C.

We may take, for instance, 5 × 11, 7 × 3 × 2, and 2 × 2 × 2 × 5, which will give 55, 42 and 40, for the numbers of the teeth that are to work in the pinions, where the numbers are 7, 5 and 11. The angular velocity of A will thus be to the angular velocity of *d* in the ratio required.

138. The number of teeth in a wheel, and in a pinion which work together, should be prime to one another, that their coincidences may not recur in the same order after every revolution.

The numbers expressing the ratio of the angular velocity of the wheel to that of the pinion, may sometimes be fractional, and may consist of so many places, when the fractions are taken away, that, to make the motions exactly in their ratio, would require more wheels, and a greater number of teeth, than can conveniently be allowed. It is then of consequence to find two numbers that will express the ratio, not exactly, but more nearly, than can be done by smaller numbers. The method of finding such numbers, is explained in the Elements of Algebra. See EULER, *Elem. d'Algèbre*, tom. II. *Additions*, § 19.; also WOOD's *Algebra*, § 371.

Of the Pulley.

139. The pulley is a wheel moveable on an axis with a groove cut in its circumference, round which a cord passes. The axis of the pulley is either fixed or moveable. When fixed, the pulley gives no mechanical advantage, but serves merely to change the direction of the power applied to the cord which passes over it. When the axis of the pulley is moveable, one end of the cord is made fast to a fixed point; the power is applied to the other, and the weight hangs by the block in which the axis of the pulley is fixed. By a single pulley of this construction, a force is enabled to balance another twice as great.

Various combinations of pulleys give various degrees of advantage; but in every case there is an equilibrium, when the spaces that would be passed over by the power and the weight in the same time, are in the inverse ratio of their quantities.

It is obvious, that the pulley is reducible to the lever of the second kind, or, still more directly, to the case, § 128. of a body supported by two props.

The combination of pulleys that has the most simplicity, is that in which a number of pulleys hang above one another in an oblique line, each doubling

bling the power of that which is under it, so that as the number of pulleys increases in arithmetical progression, the advantage increases in geometrical progression. From being so extremely portable, and so applicable to cordage, the pulley is of all the mechanical powers the most useful at sea.

Of the Wedge.

140. The wedge is a triangular prism, made of wood or of metal. It is usual to make the prism isosceles, and the angle contained between the equal sides more acute than the others. When, instead of being isosceles, the prism is rectangular, and has one of the planes containing the right angle, placed horizontally, the wedge coincides with the inclined plane, which is sometimes considered as a mechanical power distinct from the rest.

There has been a considerable difference in the rules given by mechanical writers for determining the power of the wedge. This has arisen, from not attending sufficiently to the direction of the resistance, and to another circumstance, whether both the resisting bodies are moveable, or one of them only. The best way of considering the subject, is to resolve all the forces that act upon the wedge into parts parallel to two axes, at right angles to one another, and one of them parallel to the back of the wedge, or the side to which the power is applied, In the case of equilibrium, the opposite forces, in the direction of these axes, must be equal to one another. In this way the following theorems are easily investigated.

141. When three forces are applied perpendicularly to the three faces of a triangular prism or wedge, they will be *in equilibrio*, if their directions intersect in the same point, and if they are to one another as the lengths of the sides to which they are applied.

142. If the resistance to the motion of an isosceles wedge be parallel to the back of the wedge, there will be an equilibrium, when the power applied to the back of the wedge is to the sum of the resistances, as the thickness of the back to twice the height of the wedge; and therefore if the bodies are equal, the power will be to the resistance of either of them, as half the base of the wedge to the height of it.

If one of the resisting bodies only is moveable, the power will be to the resistance as the base of the wedge to its height.

The more acute the wedge, the greater its power, or the smaller the force required to overcome a given resistance.

143. When the wedge becomes an inclined

plane, with its base horizontal, if the direction of the power be parallel to the plane, there will be an equilibrium when the power is to the resistance, as the height of the plane to its length, or as the sine of its inclination to the radius.

144. Universally there will be an equilibrium in the case of the inclined plane, when the power is to the resistance, as the sine of the inclination of the plane to the cosine of the angle which the direction of the power makes with the plane.

BOSSUT, Art. 263. *Comment. Petrop.* tom. II. p. 282.

A wedge may have the form of a pyramid, as well as of a prism. In the case of a pyramidal wedge, the forces that act must be resolved according to three axes, by which three equations will be obtained.

145. In a pyramidal wedge, where the triangular faces are inclined, at the same angle i, to the base, to produce an equilibrium, the power must be to the sum of the resistances, as the cosine of i to the radius.

Piercing instruments are all reducible to wedges of this kind, Nails, Bayonets, Stakes, Piles, &c. The resistance in some of these increases with the surface to which it is applied. Though a wedge be of a conical form, it is comprehended under the preceding theorem.

The common wedge is generally employed for cleaving, and otherwise overcoming the force of cohesion in bodies: it may sometimes be used with advantage for raising great weights to a small height.

All cutting instruments may be referred to it, Knives, Chissels, Scissars, the Teeth of Animals, &c. A Saw is a series of wedges, on which the motion impressed is oblique to the resistance. A Wimble is a wedge to which a circular motion is given.

The power applied to the wedge, when great resistance is to be overcome, is usually percussion; and almost the only instance in which it is used for the purpose of equilibrium, is in the construction of *arches*, built of truncated wedges. As the consideration of this equilibrium would lead to too long a digression in this place, it will be treated of in an Appendix.

Of the Screw.

146. The screw is a spiral groove or thread, winding round a cylinder, so as to cut all the lines drawn on its surface parallel to its axis, at the same angle. The spiral may be either on the convex or concave surface of the cylinder, and the screw is called accordingly, either the *exterior* or the *interior* screw.

The screw is properly referred to the species of wedge called the Inclined Plane, and its principle, like that of the wedge, is easily reduced to the composition of forces.

147. If the power be applied parallel to the base of the screw, and perpendicular to the radius of the cylinder; and if the weight press perpendicularly on the axis, an equilibrium is produced when the power is to the resistance, as the distance between two threads of the screw to the circumference described by the point to which the power is applied.

Muschenbroek, § 482. Bossut, § 284. S'Gravesande, § 281.

The screw is of great use for compressing bodies. A kind of percussion is sometimes added to it, as in the apparatus for coining. The great attrition or friction which takes place in the screw, is useful by retaining it in the state to which it has been once brought, and continuing the effect after the power is removed. The screw is also applied, with much advantage, to raise great weights to a small height, and support them in that situation.

The use of the screw to raise water, in the manner invented by Archimedes, is a very remarkable application of it. This screw is usually formed by a spiral tube winding round a cylinder, and it may be applied to raise any body that can pass within the tube as well as a fluid. If a screw thus formed be placed obliquely, so as to make, with the vertical, an angle equal to that which the spiral makes with the lines parallel to the axis, there will be in each turn of the spiral a part parallel to the horizon, where, if a body were placed, it would be at rest. If, then, the screw be turned, the body will ascend, because the part of the screw behind it becomes more inclined than the part before it, so that the body is urged forward, and consequently ascends.

If the screw used in this manner, be turned with great rapidity, the body may acquire a centrifugal force so great as to overcome its gravity; in which case it will descend.

On the subject of **Archimedes's** Screw, see **Bernouilli** *Hydrodynamica*, Sect. IX. § 26. Also **Euler**, *Nov. Com. Petrop.* v. p. 259, &c. **Hennert.** *sur la Vie* **d'Archimede.**

The screw is also employed in the division of mathematical instruments, and in *reading off* from them. The contrivance known by the name of the Micrometer Screw, is used for measuring angles with great exactness. In **Ramsden's** Dividing Instrument, the screw is applied for the same purpose.

Of the Funicular Machine.

148. If a body fixed to two or more ropes, is sustained by powers which act by means of those ropes, this assemblage is called the Funicular or Rope Machine.

See **Varignon**, vol. I. p. 93, &c.; also **Bossut**, *Mechanique* art. 120.

149. If a body is sustained by two forces acting at the ends of two ropes, these forces will be in the inverse ratio of the sines of the angles which their directions make with the direction of the weight; or directly as the secants of the angles which their directions make with the horizon.

Hence the sum of the powers is to the weight, as the sum of the sines of the angles which the powers make with the direction of the weight, to the sine of the angle which the powers make with one another.

150. If the body is sustained by more than two cords, the powers, by the resolution of forces, may always be reduced to two.

151. If any number of points be given either in the same or in different planes, and if from their common centre of gravity, lines be drawn to every one of the given points; the forces which draw in the direction of these lines, and are proportional to them, will be *in equilibrio.*

Hugenii, *Opera varia,* tom. i. p. 288.

It follows from the above propositions, that a small power may be able to raise a very great weight to a small height; and also, that if a rope is stretched horizontally between two points, its own weight will prevent it from becoming perfectly streight, whatever force be employed to stretch it. The curve which a flexible chain or cord forms, when suspended by its two extremities, and either hanging freely, or having any degree of tension applied to it, is called the Catenaria.

Sect.

SECT. III.

OF FRICTION.

152. BESIDES the resistance which the mechanical powers, and the machines compounded of them, are intended to overcome, an impediment to their motion is always observed when the moving parts are in contact with one another. This impediment to motion is called Friction.

> The subject of friction is treated by **MUSCHENBROEK**, § 510,—533. **BOSSUT**, § 248,—283. **PRONY**, *Arch. Hydraulique*, Sect. v. **BREWSTER**, in his Appendix to **FERGUSON**'s *Mechanics*, vol. II. p. 334. **GREGORY**'s *Mechanics*, vol. II. § 24. &c.

153. When a heavy body is at rest on a horizontal plane, it is not *in equilibrio*, or ready to obey the least impulse, in a direction parallel to the plane. A force must be applied to bring it into that state, with respect to any given direction; and this force in the measure of the friction.

154. Friction destroys, but never generates motion, and in this is unlike gravity, or any of the

forces hitherto considered, which, if they retard motion in one direction, always accelerate it in the opposite. The force of friction, therefore, violates the law of continuity, and cannot be accurately expressed by any geometrical line, or any algebraic formula.

> The retardation which friction opposes to motion, is nearly uniform, or the same for all velocities. COULOMB, in a series of experiments made with great accuracy, and on a large scale, found that bodies sliding on a plane, on which they were moved by a constant force of *traction*, were uniformly accelerated, which could not have been, if the impeding as well as the impelling force had not been constant. *Journal de Phys.* tom. XXVII. p. 204, &c. COULOMB's experiments received the prize of the Academy of Sciences in 1781, and are published in the tenth volume of the *Mémoires presentés.* An abstract of them is given in the *Journal de Phys.* above quoted, and also in PRONY's *Arch. Hyd.* Sect. v. § 1089, &c.
>
> Motions retarded by friction, are therefore subject to the laws explained, Articles 86 and 87; only that the retarding can never be converted into an accelerating force, so that the formulas of those articles cannot be applied in their full extent.

155. The force of friction is the greater, the greater the roughness or asperity of the surfaces moving on one another; it is also the greater, the greater the power by which these surfaces are

pressed together; but it is very little affected by their extent.

That the quantity of friction does not depend on the extent of the surfaces that rub on one another, was affirmed by **Amontons**, and proved by a variety of experiments. *Mém. Acad. de Sciences*, 1699, p. 208. The fact, however, has been since questioned, particularly by **Lambert**, *Mém. de Berlin*, 1772; but the experiments of **Coulomb**, though they have pointed out certain exceptions, as those of some other writers have done, have shewn, that it holds in general, and that in practical mechanics, the ratio of the friction to the pressure, may be regarded as constant, whatever be the extent of the surfaces that touch. *Journal de Phys.* vol. xxvii. p. 296. at the end.

From some very accurate experiments, Mr **Vince** concluded, that friction increases in a less ratio than that of the pressure. *Trans.* 1785, p. 172. **Bossut** has remarked the same, when the weights become very great, and gives as a proof, the small inclination of the planks on which a ship slides when it is launched. *Mechanique*, § 256.

156. Friction may be distinguished into two kinds, that of Sliding, and that of Rolling bodies. The force of the latter is very small compared with that of the former.

Though the friction of rolling is small in respect of that of sliding, great force can be given it by the application of pressure. In the passing of metals between rollers or cylinders, strongly compressed by springs, the friction is that of rolling, yet it is sufficient to communicate the motion of one of the cylinders to the plate of metal between them, in opposition to the great force which resists the extension of the plate.

157. The preceding facts appear to prove, that friction arises from the asperities at the surfaces of bodies, the eminences of the one sinking into the cavities of the other, so that a certain force, when it acts parallel to the surfaces, is necessary to disengage the parts, by forcing the one body to rise over the other, or by bending and abrading the eminences which oppose the motion.

Muschenbroek, § 512. Euler, *Mém. de Berlin*, 1748, p. 122. Coulomb, *Mém. Presentés*, tom. x. p. 164, &c.

The difference between the friction of rubbing and rolling bodies, seems to arise from this, that the asperities do not require to be bent down or abraded in the latter case; the eminences being lifted out of the cavities.

When the rising up of the one body over the other is completely prevented, the friction becomes extremely intense, as appears in what may be accounted an extreme case, the drawing of wire, tubes, &c. where the extension of the metal is the effect of the friction acting all round.

In bodies sliding on one another, the surfaces may be held together by the attraction of cohesion, so that the friction is not always diminished in proportion as the polish of the surfaces is increased. Professor Leslie is of opinion, that the phenomena of friction are not fully explained by these causes, and that they arise chiefly from a tremulous or vibratory motion in the parts of bodies. *Inquiry into the Propagation of Heat*, p. 299 to 302.

158. The distance to which a given body will be moved by percussion, in opposition to friction, is as the square of the velocity communicated to it.

This follows from § 99. Dynamics; friction being a force by which motion is uniformly retarded.

Some paradoxical appearances are explained on this principle. When a carpenter would drive the handle of an ax into the head, he strikes against the handle after it is slightly inserted into the head, holding it loosely in his hand, without any firm support, and the head perhaps downward; at each blow the handle is driven farther than if the blow had been given to the head itself. The reason is, that the handle is the lighter body, so that the velocity which a blow communicates to it is greater than that which the same blow would communicate to the head. The friction is of consequence more effectually overcome.

Thus, too, a nail is driven by a blow of no great force, into a piece of wood where the mere friction is

is sufficient to retain it against a great force applied to draw it out.

The same thing is exemplified in the method now practised, of raising a stone by an iron plug, driven into a circular hole cut in the stone. A few blows of a small hammer are sufficient to fix the plug, so that it will serve not only to suspend the stone, though of several hundred-weight, but to draw it out of the earth, in which it has lain perhaps half-buried. Such is the effect of percussion, that a few blows of a hammer, given obliquely to the plug, are sufficient to disengage it from the stone. The stones on which this experiment has been made have always possessed great toughness and induration.

159. When motion begins, the intensity of friction diminishes; it does not, however, change afterwards as the velocity changes, but continues, as already said, to retard with a uniform force.

Euler makes the friction to be reduced to one-half, when the body is actually put in motion. *Sur le Frottement des Corps Solides*, *Mém. Acad. de Berlin*, 1748, p. 122, &c. § 13. The reduction appears in some cases to be much greater than this. Coulomb found the friction of wood sliding on wood to become less when the body began to move, than it had been the instant before, in the ratio nearly of 2 to 9. Its intensity afterwards did not change. Prony, *Arch. Hydraulique*, 1173, No. 2.

160. Friction may be measured by finding the force necessary to bring a body resting on a horizontal plane into such a state, that it is ready to move on the application of the least force (§ 153.): it may also be measured by placing the body on a plane of variable inclination, and increasing that inclination till the body begin to slide. As the radius to the tangent of that inclination, so the weight of the body to its friction on the horizontal plane.

If the weight be W, and the inclination of the plane, when the body begins to slide, i, the friction $= W \times \tan i$.

161. Time is often required for friction to come to its bearing, or to attain its maximum, and in this respect, different substances differ much from one another.

I. **Coulomb** found, that in wood sliding on wood, without grease, the friction at first increased, but in a minute or two came to a limit, which it did not afterwards exceed. Oak, for example, sliding on oak, though the pressure was varied from 74 lb. to 2474, had a friction, after a minute, always nearly 44 in the hundred. On diminishing the surface as much as possible, the friction was reduced no lower than 41½ *per cent.*

II. He found also, that when metal slides on metal, the friction is proportional to the pressure; in iron on iron, 28½ in the hundred; in iron on brass 26, &c.; and this was the same whether the bodies were just beginning to move from rest, or had acquired any velocity whatever different from the case just mentioned, of wood rubbing on wood.

III. When heterogeneous bodies were made to slide on one another, as wood on metal, the friction increased slowly with the time, and did not come to its maximum in less than four or five days. Iron against oak, after 10″, gave the force of friction = 7⅘ *per cent.*; at the end of four days, it amounted nearly to one-fifth. In heterogeneous substances, too, the friction increases sensibly with the velocity, and follows nearly an arithmetical, while the velocity follows a geometrical progression.

IV. When the surfaces are smeared with unctuous substances, the friction is diminished; but time is necessary for the attainment of the maximum. In a set of experiments, oak rubbing on oak, the surfaces greased with tallow, and the weights being 1650, and 3280 pounds; in the first, at the end of six days, the friction seemed to become stationary, and was 37½ *per cent.*; in the second, the maximum arrived at the end of five days, and the friction was 47.

Another set, with brass on iron, and fresh tallow between it, was four days before the friction came to its maximum, when it was from 10 to 11 *per cent.* At the first moment it was 9 *per cent.*, so that here the increase was inconsiderable. These are some of the principal results of COULOMB's experiments, the most accurate and extensive that have been made on this matter; they shew a great difference in the laws of friction, that obtain among different substances.

Consult **Prony** and **Coulomb**, *ubi supra*.

162. Friction is diminished by the unctuous substances mentioned above. Those that are thinnest and least tenacious are the best; plumbago also, or Black Lead, as it is called, reduced to powder, and rubbed on the surfaces of wood, metal, stone, &c. serves greatly to diminish friction.

163. The effect of friction may be diminished, by drawing a body in a line inclined at a certain angle to the plane on which it rests. Thus, if the weight of a body be to its friction, on a horizontal plane, as n to 1, it will be drawn with the greatest ease in the direction which makes with that plane an angle, having for its tangent $\frac{1}{n}$.

Dan. Bernouilli, *Nov. Com. Petrop.* tom. 13. p. 244.

164. Friction is diminished, when it is converted, by means of wheels or rollers, from the friction of sliding into that of rolling bodies.

When each end of the axis of a wheel is made to turn, not in a groove, but on the circumferences of two wheels, every one moveable on its own axis, these last are called Friction Wheels, as serving very much to lessen the effects of friction.

165. The momentum of friction is diminished by friction wheels, in the ratio of the radius of the axis of any one of the wheels (they are supposed equal) to the perpendicular height of the axis that rests upon them, above the line joining their centres.

EULER treats of the advantage of friction wheels, *Mém. Acad. Berlin*, 1743, p. 144. The theorem he has investigated is not so simple as the preceding, but may be reduced to it. In the *Phil. Trans.* vol. LIII. p. 155. Mr K. FITZGERALD has described the construction of friction wheels, by which he reduced the power of friction from 425 lb. to two and three-fourths.

Rollers put under a heavy body, diminish the friction in the greatest degree possible, if they are spheres or cylinders, without any axis on which they are constrained to move. The conversion of friction from sliding to rolling is then complete.

The wheels of carriages are contrivances of the same kind: but in them the conversion above referred to is imperfect: for though the friction at the circumference of the wheel is that of rolling, at the axis, it is that of rubbing; yet as the distance from the centre of motion is small, the momentum of the friction is small in the same proportion.

166. In wheel carriages, it is more advantageous that the wheel should turn on a fixed axis, than that the axis should be moveable, and turn with the wheel, because a rotatory motion must be given to the axle in the latter case, and not in the former.

167. In wheel carriages, the plane on which they move, and the line of draught, being both horizontal, the advantage for surmounting an immoveable obstacle, of a given height, is as the square root of the radius of the wheel.

Let the whole weight to be moved be W, the radius of the wheel r; f the force which, drawing horizontally, will raise the carriage over an immoveable obstacle of the height h; then

$$f = W\sqrt{\frac{2h}{r}};$$

or $\sqrt{r} : \sqrt{2h} :: W : f$.

168. When a machine is so loaded, that it would be *in equilibrio* if there were no friction, it will not be ready to move till a part be added to the power, having its momentum equal to the momentum of the friction.

The

The friction must be considered as a given force, opposed to the power, and having a momentum proportional to its distance from the centre or axis of motion.

169. Let a lever turn on an axle of which the radius is r; let a be the length of the arm to which the power P is applied, in order to raise the weight W; and let p be the addition to be made, in order to overcome the friction, the ratio of the pressure to the friction being n to 1; then is,

$$p = \frac{r}{n\,a}(P + W).$$

P is understood to be expressed in weight.

For the application of this to the other mechanical powers, see Prony, *Arch. Hyd.* Sect. v. Bossut, *Mech.* § 261, to 283. Van Swinden, *Positiones Phys.* § 312—325.

170. The stiffness of ropes, or the force requisite to bend them, has a great analogy to friction. In different ropes, the forces requisite to bend them, are in the direct ratio of their diameters and their tensions jointly, and in the inverse ratio of the radii of the cylinders round which they are bent.

Van Swinden, *Pos. Phys.* 339. Coulomb, whose experiments ought to have great weight, holds the stiffness to follow the ratio of the section of the rope, that is, of the square, not of the simple power of its diameter. The decision of the question requires new experiments.

171. The friction of a rope that is wound round a cylinder, increases in geometrical progression, while the number of turns increases in arithmetical progression.

If the turns be represented by the numbers, 0, 1, 2, 3, 4, &c.; the resistance made by the rope may be represented by the numbers 1, 2, 4, 8, 16, &c.

Hence the great use of this kind of resistance to stop motion quickly, and to make fast one end of a rope when a great force or strain is to be applied to the other.

172. Though friction destroys motion, and generates none, it is of essential use in mechanics. It is the cause of stability in the structure of machines; and is necessary to the exertion of the force of animals.

A nail, a screw, or a bolt, could give no firmness to the parts of a machine, or of any other structure, without friction. Animals could not walk, or exert their force any how, without the support which it affords. Nothing would have any stability but in the lowest possible situation, and an arch that could sustain the greatest load when properly distributed, might be thrown down by the weight of a single ounce, if not placed with mathematical exactness at the very point it ought to occupy.

SECT. IV.

OF MECHANICAL AGENTS.

173. A FUNDAMENTAL distinction among mechanical agents, or the powers which put bodies in motion, consists in this, that in some the intensity of their action, or the acceleration they produce in a given time, is the same whether the body acted on be at rest or in motion; in others, it is greatest when the body acted on is at rest, and becomes less as its velocity increases.

Gravity is the only force that is certainly known to act with equal intensity on bodies in motion and at rest. Magnetism has probably the same property. Every other power acts more forcibly on a body at rest than on one that has already acquired motion in the direction in which it acts. This happens with respect to the elasticity of springs, the impulse of fluids, and the strength of animals, the only powers, except gravity, which are employed to put machines in motion. The knowledge of the laws, according to which the action of these powers diminishes, as the motion they communicate increases, is of great importance in mechanics.

174. The absolute elasticity of a spring is nearly proportional to its distance from its state of quiescence, or that at which its elasticity does not act; but the power of the same spring to impel a body, must depend not only on its absolute force, but also on the rate at which it overtakes that body.

An arrow, for example, is impelled by the string of a bow, not simply by a force proportional to the distance of the string from the state in which it is at rest (to which the absolute elasticity of the bow is probably proportional) but by a force that is less as the difference between the velocity of the string and of the arrow diminishes.

This subject, however, has not been sufficiently examined.

A spring is sometimes made to act with uniform force by lengthening the lever to which it is applied in the same proportion as its tension becomes less, or as its distance from its state of quiescence diminishes. This is done in the case of a watch-spring. The velocity acquired by the body impelled, is here so small that it may be neglected.

175. When a moving body, whether solid or fluid, acts upon another, the motion communicated is less, the less the difference of their velocities.

The determination of the law observed by this action, in the case of fluids, belongs to Hydrodynamics, and will be explained under that head. There remains, therefore, to be considered here only the law of animal force.

176. The strength of men, and of all animals, is most powerful when directed against a resistance that is at rest: when the resistance is overcome, and when the animal is in motion, its force is diminished; lastly, with a certain velocity, the animal can do no work, and can only keep up the motion of its own body.

A formula having the three properties just mentioned, will afford an approximation to the law of animal force. Let P be the weight which the animal exerting itself to the utmost, or at a *dead pull*, is just able to overcome; W any other weight with which it is actually loaded; and v the velocity with which it moves when so loaded; c the velocity at which the power of drawing or carrying a load entirely ceases; then $W = P\left(1 - \frac{v}{c}\right)$ is an equation that has all the three conditions mentioned above.

Not only, however, has the formula $P\left(1 - \frac{v}{c}\right)$ these conditions, but the square of it has the same, or, indeed, any function of it which vanishes when $1 - \frac{v}{c}$ vanishes, that is, when $v = c$. We are left, then,

then, at liberty to choose any of these functions, and would assume the formula above as the simplest, if another condition did not seem necessary to be included. It is certain, that in all cases, when v approaches to c, or when the speed becomes great, a small variation in the weight is accompanied with a great variation in the velocity. The simplest formula that corresponds to this condition, is, when $1 - \frac{v}{c}$ is raised to the square.

177. Therefore, till experience has led to a more accurate result, we may suppose the strength of animals to follow the law expressed by the formula, $W = P\left(1 - \frac{v}{c}\right)^2$.

This equation, supposing W and v variable, is an equation to a parabola, the construction of which will serve to represent this law more clearly to the imagination.

A formula for expressing the law of animal action, was first proposed by **Euler**, in a Dissertation on the Force of Oars. *Mém. Acad. de Berlin*, 1747. That which he employed was $W = P\left(1 - \frac{v^2}{c^2}\right)$, different from both those we have mentioned, but a function of the first, and one that becomes 0, when $v = c$. **Euler**, however, changed this to another, *Mém. Acad. de Berlin*, 1752, and *Nov. Com. Petrop.* VIII. p. 244., the same that we have adopted. He appears to have done so merely on account of the analogy thus preserved between the action of animals and of fluids. The physical fact mentioned above, is a better reason for the preference.

178. The *effect* of animal force, then, or the quantity of work done in a given time, will be proportional to $W v$, or to $P v \left(1 - \frac{v}{c}\right)^2$, and will be a *maximum* when $v = \frac{c}{3}$, and $W = \frac{4P}{9}$, that is, when the animal moves with the one-third of the speed with which it is able only to move itself, and is loaded with four-ninths of the greatest load it is able to put in motion.

The quantities P and c can only be determined by experience, and as they must differ for different individuals, an average estimation of them is all that can be obtained. Even that average is but imperfectly known; EULER supposes, that for the work of men, P may be taken = 60 lb. and $c = 6$ feet *per* second, or a little more than four miles an hour.

A man, according to this estimate, when working to the greatest advantage, should carry a load of 27 lb., and walk at the rate of two feet in a second, or a mile and one-third an hour.

A horse, according to **Desaguliers**, drawing a weight out of a well, over a pulley, can raise 200 lb. for eight hours together, at the rate of two miles and a quarter an hour. Supposing the horse, in this case, to work to the greatest advantage,

$$P = \frac{200 \times 9}{4} = 450, \text{ and } c = 2.25 \times 3 = 6\tfrac{3}{4} \text{ miles}$$

per hour.

This estimate seems to give too high a value to P. It will suit better with general experience to make $P = 420$, and $c = 7$.

When a horse's work is estimated by the load he draws in a cart or waggon, a great reduction must be made, in order to compare the force he exerts with that which is necessary for raising a weight, by drawing it over a pulley. Though accurate experiments on the friction of wheel carriages are wanting, we probably shall not err much in supposing the friction, on a road, and with a carriage of the ordinary construction, to amount to a twelfth part of the load. If, then, a horse draws a ton in a cart, which a strong horse will continue to do for several hours together, we must suppose his action the same as if he raised up the twelfth part of a ton, (2240 lb.), or 186 lb. perpendicularly against the force of gravity. To raise a weight of 186 lb. therefore, at the rate of two miles, or two miles and a half an hour, (that is 2.9, or 3.6 feet *per* second),

cond), may be taken as the average work of a strong draught horse in good condition.

A different view of the manner of estimating animal force, has been taken by Coulomb, *Mém de l'Inst. Nat.* tom. II. p. 380, &c.

179. It appears to be a certain fact, that when a man carries only his own weight, the quantity of his action, that is, the height he is able to ascend in a given time, multiplied into his weight, is greater than when he carries any additional load; and Coulomb thinks it probable, that this diminution of action, is in proportion to the additional load carried. Now it appeared from his experiments, that when a man carried a load equal to his own weight, his action was reduced nearly one-half; and, therefore, supposing the reduction always proportional to the load, if w be the weight of the man's body, l an additional load, which he is made to carry, H the height to which he ascends in a given time, when walking freely, and h the height to which he ascends in the same time with the load l; then his action in the latter case, or $(w+l)\,h$, is reduced $w\,\mathrm{H}\left(1-\frac{l}{2w}\right)$; and therefore also $h = \frac{w\,\mathrm{H}\left(1-\frac{l}{2w}\right)}{w+l}$.

Suppose that a man is loaded with one-fourth of his own weight; then $h = \dfrac{w\,\mathrm{H}\,(1 - \frac{1}{8})}{w\,(1 + \frac{1}{4})}$

$$= \mathrm{H}\left(\frac{1 - \frac{1}{8}}{1 + \frac{1}{4}}\right) = \mathrm{H}\,(.699).$$

The value of H is deduced from the ascent of the Peak of Teneriffe. BORDA, accompanied by eight men, on foot, ascended in the first day ($7^{\mathrm{h}}\ 45^{\mathrm{m}}$), to the height of 2923 metres, or 9593 feet. This was at the rate of 1225 feet in an hour. Had each of the men carried a load equal to the fourth part of his weight, they would only have ascended at the rate of 857 feet an hour.

When $1 - \dfrac{l}{2\,w} = 0$, or $l = 2\,w$, the height $h = 0$.

With a load equal to twice a man's weight, he could not ascend.

180. The strength of a man being supposed to follow the law now laid down, its greatest effect in raising a weight, will be when the weight of the man is to that of his load as 1 to $-1 + \sqrt{3}$, or nearly as 4 to 3.

Because $h = \frac{Hw\left(1 - \frac{l}{2w}\right)}{w + l}$, $lh = \frac{Hwl\left(1 - \frac{l}{2w}\right)}{w + l}$; now lh, or the weight multiplied into the height to which it is raised, is the measure of the effect, or of the work done, which, therefore, will be a maximum when the last formula is so, that is, when $l = w(-1 + \sqrt{3})$.

If in the equation $h = \frac{Hw\left(1 - \frac{l}{2w}\right)}{w + l}$, we suppose h and l to be variable, the other quantities being constant, the locus of the equation is a hyperbola, which may be easily constructed.

The theorems just given only differ from COULOMB's, by being somewhat simpler, and free from all reference to any particular measure of length or of weight.

On this subject, however, many more experiments are wanting; to determine, for instance, the friction of wheel carriages; the difference of the exertion required to walk on a horizontal plane, and on one of a given declivity; the quantity of work done in a given time by the same animal, carrying different loads. The difference between the effective exertion of a man's strength when he moves along with his load, and when he stands, as in turning a wheel, or sits, as in rowing a boat, &c. &c.

SECT. V.

MOTION OF MACHINES.

181. MACHINES either work with a variable or a uniform velocity. If the moving power is of the kind, that when the motion begins, diminishes in the intensity of its action, the machine, after a little time, will acquire a uniform velocity. If, however, the moving power be one that acts always with the same force, and if the resistance is also uniform, the machine will be continually accelerated. But as a motion continually accelerated is incompatible with the purposes of machinery, it is either contrived, that, by an increased resistance, the motion shall become uniform, or that the moving power shall at intervals cease altogether, or become a retarding force,—so that the velocity of the machine, though continually changing, may be confined within certain limits.

We shall begin with considering some of the simpler machines of this latter class, when gravity is the moving power.

182. Let A and B be two given weights, applied to the ends of the arms of a lever, of which the lengths are a and b, and let $A \times a$ be greater than $B \times b$, so that A may preponderate; the space over which B will be raised in one second, (g being the velocity which a heavy body acquires in the first second of its descent), is $\frac{(A a - B b) b}{A a^2 + B b^2} \times \frac{1}{2} g$.

Call x the velocity of b, then $\frac{a x}{b}$ will be the velocity of A. If the bodies were free to fall by their own gravity, the velocity of each would be g; therefore the velocity lost by A is $g - \frac{a x}{b}$; and the velocity gained by B is $g + x$; now the momenta of these must be equal by § 117, that is, $A a \left(g - \frac{a x}{b}\right) = B b (g + x)$; and hence x, or the velocity of B, is found, and the half of it is the space over which B is raised in one second.

Hence the time in which B will be raised over a given space h, being equal to as many seconds as h contains the above expression, is =

$$\frac{A a^2 + B b^2}{(A a - B b) b} \times \frac{2 h}{g}.$$

It is evident, that if A is given, as also B and b, and if the length of the arm a be alone supposed variable,

able, the time thus computed must admit of a minimum, as it would be infinite if a were so short, that $A\,a = B\,b$; and it would also be infinite, if a were infinitely long. There must be a value of a, therefore, that will give the time of ascent the least, or the angular velocity of the lever the greatest; and it will be found by the preceding formula.

183. The same things, therefore, being supposed as in the last proposition, the angular velocity of the lever will be the greatest possible, when

$$a = \frac{B\,b}{A} \pm \frac{b\sqrt{B^2 + AB}}{A}.$$

If the bodies A and B are equal,

$$a = b + b\sqrt{2} = b\,(1 + \sqrt{2}).$$

In the investigation of the proposition in § 182., the use of M. D'Alembert's Principle of the Distribution of Motion is exemplified. The problem, however, has been resolved by other means, though less satisfactory and less generally applicable.

Euler, *de Machinarum usu maxime lucroso, Com. Petrop.* (1738) p. 67. Prop. 2. Also Blake, *Phil. Trans.* 1751, p. 1., &c.

We have abstracted here from the inertia and weight of the lever, that the investigation might be the more elementary. Euler has extended his inquiry to the wheel and axle, and other machines impelled by gravity.

184. A state of uniform motion is sometimes produced in machines where gravity is the moving power, by means of a resistance, so constituted, as to increase with the velocity: when the velocity has reached a certain quantity, the retarding force becomes equal to the accelerating, and the velocity of the machine remains uniform.

This is exemplified in the motion of a jack; the resistance that the fly makes, increases with its velocity, and after a certain time, prevents all farther acceleration.

A more refined artifice for producing a uniform motion in a machine impelled by gravity, is employed in the pendulum clock.

In that machine, a pendulum, by forcing the end of a lever at each vibration, against a tooth of a wheel, put in motion by a weight, prevents the acceleration of that weight, and at the same time receives an impulse just sufficient to continue its motion. The exact adjustment of these forces, is one of the nicest problems in practical Mechanics.

In a watch, where the moving power is the elasticity of a spring, the uniformity of the motion is, on similar principles, produced by the vibrations of a balance.

185. When the momentum of the power applied to a machine, is greater than the momentum of

the resistance, the machine is put in motion, and if the power be one that acts more forcibly on bodies at rest than in motion, its action is of course diminished, and the acceleration of the machine in the second instant is less than in the first. This diminution of the accelerating force continues to increase, till the instant when that force becomes equal to the resistance, or till the velocity generated every instant by the one, be just equal to the velocity destroyed every instant by the other. The acceleration then ceases, and the motion of the machine becomes uniform.

The process by which a machine is thus brought to its *permanent state* of working, is most rapid in the beginning, and becomes slower as it advances. Indeed, if the laws of physical action admitted of mathematical exactness, or rather, if our senses enabled us to trace their progress with perfect accuracy, it would be found that a state of uniform motion is never fully acquired by any machine, and that it cannot be acquired in less than an infinite time. The curve of which the abscissæ represent the time, and the ordinates the velocity of the machine, can never pass into a straight line; but it may have, for an assymptote, a straight line parallel to its axis, to which it may soon approach so near that they can no longer be distinguished.

On the motion of machines, see Euler, *de Machinis in genere*, *Nov. Com. Petrop.* tom. III. (1750, 1751), p. 254.,

p. 254., &c. Also *Principia Theoriæ Machinarum*, by the same Author, *Nov. Com. Petrop.* tom. VIII. (1760, 1761), p. 230., &c.

186. When a machine attains its state of uniform motion, the momentum of the power is equal to that of the resistance, and is the same that would be *in equilibrio* with the resistance, if there were no motion at all.

If it is by a stream of water that the machine is driven, the stream is, by the supposition, at first much more than able to overcome the resistance, and the machine accelerates till the relative velocity of the water and the wheel is so small, that the force of the water is no greater than is sufficient to balance the resistance. The case is nearly the same whatever be the moving power, provided its action diminish with the velocity it communicates.

187. In all machines, the work done is to be estimated not merely from the quantity of resistance overcome, but from the quantity overcome in a given time, or, which is the same, from the quantity of the resistance multiplied into the velocity communicated.

The resistance is supposed here to be expressed by weight, and the effect by that weight multiplied into the velocity with which it is raised.

As this is also the expression for the momentum of the resistance, and as that momentum is equal, when the motion is uniform, to the momentum of the power, this last, or the product of the moving force into its velocity, is also the measure of the effect.

188. In all machines that work with a uniform motion, there is a certain velocity, and a certain load, that yield the greatest *effect*, and which are therefore more advantageous than any other.

If the machine is loaded heavily, though the resistance overcome may be great, it may be overcome so slowly, that the total effect shall be but small. And, again, if the machine is loaded very lightly, it will give great velocity to the load, though, from the smallness of its quantity, the effect will still be inconsiderable. Between these two loads there must be some intermediate one, that will make the effect the greatest possible.

If the moving power observe the same law that has been already ascribed to animal force, viz. $W = P\left(1 - \frac{v}{c}\right)^2$, (which will be afterwards shewn to be the law that regulates the force, of fluids in motion), the effect will be expressed by Wv, or by $Pv\left(1 - \frac{v}{c}\right)^2$, which is a maxi-

mum when $W = \frac{4P}{9}$, and when $v = \frac{c}{3}$. The resistance should be $\frac{4}{9}$ of that which the power, when fully exerted, is just able to balance, or of the resistance that is sufficient to reduce the machine to rest; and the velocity of the part of the machine to which the power is applied, should be one-third of the greatest velocity of the power.

189. If, therefore, the moving power in any machine follow the law expressed by the equation $W = P\left(1 - \frac{v}{c}\right)^2$; and if the load or resistance that is just able to keep the machine at rest, or to prevent its motion altogether, be found by experiment; then if the load be reduced to $\frac{4}{9}$ of this quantity, the effect of the machine will be the greatest possible.

190. The moving power and the resistance being both given, other things remaining as above, if a machine be so constructed that the velocity of the point to which the power is applied, be to the velocity of the point to which the resistance is applied as 9 R to 4 P, the machine will work to the greatest possible advantage.

The velocities of the points just mentioned are capable of being adjusted to any given ratio, on principles that have already been explained.

Under the name of the Load, we suppose the friction of the parts of the machine to be comprehended, which must therefore be determined by experiment.

191. Care should be taken to give to every machine the greatest possible regularity in its motion.

The contrivance known by the name of a Fly, is one of the best adapted for this purpose. When the impelling power is subject to alternate intention and remission, the inertia of a heavy wheel, by preserving the velocity it has acquired, tends to lessen the irregularity resulting from these causes.

The effect of a fly, to remove irregularity in the motion of a machine, its weight and diameter being given, is proportional to its velocity.

192. In wheel carriages, the equability of the motion is even of greater importance than in hydraulic engines; the sudden checks to motion having a more pernicious effect on an animate, than it can have on an inanimate body.

The improvement of roads, and the diminution of friction, is, in such cases, of great importance; but that which corresponds most to the advantage derived from the use of a fly, is the supporting of the weight by springs, or even giving elasticity to the parts by which the cattle are yoked to the carriage. The sudden action of the resistance, and the jerks which are the consequences of it, are thus taken away. The greater the velocity the carriage moves with, the more advantage is derived from this application of elastic force.

SECT. VI.

OF THE DESCENT OF HEAVY BODIES ON PLANE AND CURVED SURFACES.

193. THE force which accelerates the motion of a heavy body on an inclined plane, is to the force of gravity, as the sine of the inclination of the plane to the radius, or as the height of the plane to its length.

If $f=$ force accelerating the body on an inclined plane, of which the inclination is i, and if $g=$ force of gravity, $f=g \sin i$. Hence the motion of a body on an inclined plane, is a motion uniformly accelerated.

S'GRAVESANDE, § 382.; MUSCHENBROEK, § 608.: GALILEO, *Dial.* III. *Opere*, tom. III. p. 106, &c.

194. If two bodies begin to descend from rest, and from the same point, the one on an inclined plane, and the other falling freely to the ground, their velocities at all equal heights above the surface will be equal.

> Hence the velocity acquired by a body in falling from rest through a given height, is the same, whether it fall freely, or descend on a plane any how inclined.

195. The space through which a body will descend on an inclined plane, is to the space through which it would fall freely in the same time, as the sine of the inclination of the plane to the radius.

> The diameter of a circle perpendicular to the horizon, and any chord terminating at either extremity of that diameter, are fallen through in the same time.

> The velocities which bodies acquire by descending along chords of the same circle, are as the lengths of those chords.

196. If a body descend over a series of inclined planes, at each of the angular points, where it passes from one plane to another, it loses a part of the velocity it had acquired, proportional to the versed sine of the inclination of the planes.

If v be the velocity it has acquired when it comes to any angle φ, $v \times \text{vers.}\,\varphi$ is the velocity lost.

197. Any angle being given, it may be divided into angles, so small, that the sum of the versed sines of these angles shall be less than any given magnitude.

Hence the number of planes may be so increased, and their inclination to one another so diminished, that though the change of direction between the first and the last be ever so great, the loss of velocity in the descending body shall be less than any given quantity.

Hence, if the body descend in a curve, it suffers no loss of velocity.

D'Alembert, *Traité de Dynamique*, § 41.

This is true of the motion of a *pendulum*, that is, of a body hanging by a thread, and swinging backwards and forwards about a fixed point.

198. In a pendulum vibrating in a circle, the force which accelerates it when it approaches the lowest point, is nearly proportional to its distance from that point.

Hence the small vibrations of the same pendulum are nearly *isocronous*.

199. If a pendulum vibrate in a circle, and if g be the velocity acquired by a heavy body descending freely, in one second, the square root of g is to the square root of the length of the pendulum, as the number which expresses the ratio of the circumference of a circle to its diameter, is to the time of the least vibration of the pendulum, expressed in seconds.

Let l be the length of a pendulum, π the number that denotes the circumference of a circle, of which the diameter is 1, and $t =$ the time of one vibration of the pendulum in seconds,

$$t = \pi\sqrt{\frac{l}{g}}.$$

As the weight of the body does not enter into the expression of the time, the vibrations of a pendulum are the same whatever be its weight.

Hence the times of the vibrations of pendulums are as the square roots of their lengths.

If $t = 1$, $l' =$ length of the second's pendulum, and

$$1 = \pi\sqrt{\frac{l}{g}}\text{ ; therefore } g = \pi^2 \times l', \text{ or } l' = \frac{g}{\pi^2}.$$

If it be required, having l', to find g, $g = \pi^2 \times l$.

By help of this last formula, g is found more exactly than can be done by direct experiment. In the latitude of London, the length of the seconds pendulum $= 39.126$ inches. Hence $g = 32.18$ feet.

The values both of g and l' are somewhat different in different latitudes. The nature of this variation will be explained in Physical Astronomy.

Centre of Oscillation.

200. When a pendulum consists of two or more bodies, or of one body, from the figure and extent of which we are not permitted to abstract, it is called a Compound Pendulum; and its centre of oscillation is a point in it, at such a distance from the centre of suspension, that a simple pendulum of a length equal to that distance, will have the same angular velocity with the compound pendulum itself.

201. The distance of the centre of oscillation of a compound pendulum, from its centre of suspension, is equal to the sum of the products of each body into the square of its distance from the centre of suspension, divided by the sum of the products of each body into the simple power of its distance from that centre.

If the bodies be A, B, C, D, &c. and their distances from the centre of suspension a, b, c, d, &c. and x the distance of the centre of oscillation from the same point, then

$$x = \frac{A a^2 + B b^2 + C c^2 + D d^2}{A a + B b + C c + D d}.$$

This value of x is deduced from D'ALEMBERT's principle, § 117.

x is also the length of the simple pendulum isocronous with the compound.

If any of the bodies are above the point of suspension, their distances from that point are accounted negative.

202. If a cylinder, of which the altitude is a, and the radius r, be suspended from its vertex, the distance of the centre of oscillation from the vertex is $\frac{2a}{3} + \frac{r}{2a}$.

If the radius $r = 0$, so that the cylinder coincides with a straight rod of inconsiderable thickness, the distance of the centre of oscillation is $\frac{2a}{3}$.

203. In a cone suspended from the vertex of which the altitude is a, and the radius of the base r, the distance of the centre of oscillation from the vertex is $\frac{4a}{5} + \frac{r^2}{5a}$.

204. If a sphere, of which the radius is n, be suspended from any point on its surface, the dis-

tance of the centre of oscillation from that point is $\frac{7r}{5}$.

WOLFII *Elementa Math.* tom. II. § 459, 460.

205. If in an axis which divides a figure into equal and similar parts, different points be taken, by which the figure is suspended, the distances of the centres of oscillation from the centre of gravity of the figure, will be reciprocally as the distances of the points of suspension from the same.

This proposition holds of solids as well as of plane figures.

206. The centres of suspension and oscillation are convertible, that is, if the centre of oscillation in any body or system of bodies be made the point of suspension, the former point of suspension will become the centre of oscillation.

The first author who treated of the pendulum and the centre of oscillation, was HUYGENS, in his *Horologium Oscillatorium*, *Opera Varia*, tom. I. p. 51, &c. See also S'GRAVESANDE, § 404, and 424. MUSCHENBROEK, § 640. BOSSUT, *Mechanique*, § 401, 405.

Heavy Bodies descending on a Cycloidal Surface.

207. If a circle DHO (fig. II.) roll along a line AB, until the point V in its circumference, which touched the line AB in A, come to touch it again in B, the curve AVB, generated by the point V, is called a *Cycloid.*

> The line AB is called the Base; DV, which bisects AB at right angles, the Axis of the Cycloid; and the circle DHO, the Generating Circle.
>
> If AC and BC be two semicycloids, or semicycloidal cheeks, each equal to the half of AVB, touching AB in A and B, and touching one another in C; and if there be fixed at C a pendulum D, hanging by a thread PTC, equal in length to the semicycloid, P in its vibrations will describe the cycloid AVB.

208. A pendulum being thus made to vibrate in a cycloid, is at every moment of its descent accelerated by a force proportional to the arch of the cycloid, between it and the lowest point.

> Hence the vibrations of this pendulum, whether great or small, are all of the same duration.

209. If l be the length of a pendulum vibrating in a cycloid, the time of a vibration, whether great or small, will be $\pi\sqrt{\frac{l}{g}}$.

210. The line in which a heavy body descends in the least time from one given point to another, not in the same horizontal plane, nor in the same vertical line with it, is an arch of a cycloid, having for its base a horizontal line drawn through the uppermost of the given points.

Hence the cycloid is called the *line of swiftest descent.*

The problem of finding the line of swiftest descent was first proposed by John Bernouilli, *Acta Eruditorum*, 1697. Though the solution of it requires the assistance of the higher geometry, more elementary demonstrations have been given by several authors, particularly by Maclaurin, *Fluxions*, vol. ii. § 574, &c.; and by Frisius, *Cosmographia*, *Introd.* § 22, and 23.

Sect.

SECT. VII.

ROTATION OF BODIES.

Rotation about a Fixed Axis.

211. LET A, B, C, D, &c. (fig. 12.) be a system of bodies in one plane, so connected as to be immoveable in respect of one another, but moveable about an axis at right angles to the plane, and passing through a given point S; let a, b, c, d, be their distances respectively from S; and let a force act on one of the bodies A, such that if A were unconnected with the system, it would receive the velocity u, in a direction at right angles to the radius a; let v be the velocity which it actually acquires when it makes a part of the system;

$$v = \frac{A a^2 u}{A a^2 + B b^2 + C c^2 + D d^2}:$$

That is, if each body be multiplied into the square of its distance from the axis of rotation, the sum of all these products, is to the product of A, into the square of its distance from the same axis, as the velocity which A would have had if it had

been

been free to move by itself, to the velocity which it has as a part of the system.

This is proved, by considering that A $(u-v)$ is the motion lost by A, and $\frac{Bbv}{a}$, $\frac{Ccv}{a}$, $\frac{Ddv}{a}$, the motions gained by B, C, and D; and that these last must, by D'ALEMBERT's principle (§ 117.), be *in equilibrio* with the former; whence the equation $Aa(u-v) = (Bb^2 + Cc^2 + Dd^2)\frac{v}{a}$, from which the value of v is obtained.

From the demonstration, it is evident, that the momentum of the system relatively to S, is found by multiplying each body into the square of its distance from S, and the sum into the velocity at A.

The angular velocity is found by dividing both sides by a; this gives

$$\frac{v}{a} = \frac{Auu}{Aa^2 + Bb^2 + Cc^2 + Dd^2}.$$

The quantity $\frac{v}{a}$ is the value of the angle described round S in one second, expressed in parts of the radius. If it is required in degrees and minutes, the above value must be multiplied by 57°.29578, the number of degrees in an arch equal to the radius.

These propositions hold not only of a system of bodies in one plane, but of a system in different planes, if we suppose perpendiculars drawn from them to the axis of rotation.

212. If φ, φ', φ'', be the angles which the radii drawn from S to B, C, D, (fig. 13.) make with the radius drawn through A, and if in this latter radius a point P be taken, distant from S by the quantity

$$x = \frac{A a^2 + B b^2 + C c^2 + D d^2}{A a + B b \cos \varphi + C c \cos \varphi' + D d \cos \varphi''},$$

the point P is such, that if an obstacle were applied there sufficient to resist the rotation of the system, no motion would be communicated to the axis at S; or, which is the same, though the axis at S were not fixed, the system would have no tendency to revolve about the point P.

The point P is called the *Centre of Percussion.*

This proposition is investigated by assuming P, drawing PO parallel to BS, and BO perpendicular to it. The momentum of B relatively to P is the same as if it were placed at O, and is therefore $B \times \frac{b.v}{a} (b - x \cos \varphi)$. The sum of the momenta computed in this way is equal to 0, and thence the value of x, or SP.

213. Any system of bodies being given, a point may be found in which if all the bodies were collected, a force applied at any distance from the

the axis, would communicate to the bodies thus collected the same angular velocity that it would have communicated to the system in its first condition.

The point thus found is called the *Centre of Gyration.*

Its distance from the axis of rotation is

$$\sqrt{\frac{A a^2 + B b^2 + C c^2 + D d^2}{A + B + C + D}}.$$

Atwood *on Rectilinear and Rotatory Motion*, p. 211, 212.

As every body may be considered as a system of physical points, it is evident that the preceding propositions are of general application. We shall give examples in the cases of the cylinder and the sphere.

214. Let ADBE (fig. 14.) be a cylinder moveable about an axis passing through C its centre; let the radius of the cylinder be given $= r$, and also its weight $=$ W; and let another given weight F hang by a cord BF, wound round the cylinder. Required, the force accelerating the point B, and the velocity which it will acquire in the first second of its motion.

The sum made by multiplying each particle of the cylinder, into the square of its distance from the axis, supposing the thickness of the cylinder, in the direction of that axis, to be m, is found by fluxions, or the quadrature of curves, to be $\frac{1}{2}\pi m r^4$, and therefore (by § 211.) the velocity acquired in the first second by the circumference of the cylinder is $= \frac{2 F r^2 . g}{2 F r^2 + \pi m r^4} = \frac{2 F . g}{2 F + \pi m r^2}$, or, (since $\pi m r^2 =$ the solidity of the cylinder), $= \frac{2 F . g}{2 F + W}$; g being the velocity acquired by a heavy body in the first second of its descent.

As $2 F + W$ to $2 F$, so the force of gravity to the force accelerating the cylinder at its surface.

215. The quantity $A a^2 + B b^2$, &c. in § 211., which is the sum of all the products made by multiplying each particle into the square of its distance from the axis of motion, is called the *momentum of the inertia* of the revolving body. It is computed for any given body, either by the method of fluxions, or the quadrature of curves.

216. Let the circle ADB (fig. 14.) represent a sphere moveable about an axis that passes through the centre C, and is perpendicular to the plane of the circle ADB; and let it be put in mo-

tion by the weight F hanging from B. The radius of the sphere r, its weight W, and that of the body F being given, it is required to find the velocity which the point B acquires in one second, or the force by which it is accelerated.

The momentum of inertia computed for the sphere, is $\frac{4}{15}\pi r^5$, and therefore the accelerating force at B is $\frac{F r^2 . g}{F r^2 + \frac{4}{15}\pi r^5} = \frac{F . g}{F + \frac{4}{15}\pi r^3}$. As $2\pi r^3$ is the solidity of the sphere, if W be substituted for it, the above expression becomes $\frac{F . g}{F + \frac{2}{15} W}$.

Therefore $F + \frac{2}{15} W$ is to F, as the force of gravity to the accelerating force at B, the equator of the sphere.

217. If the motion, instead of being produced by a body hanging from the sphere, and partaking of its velocity, were communicated by the impulse of F, moving with a given velocity c, in the direction of the tangent at B, and striking against the radius produced, the velocity communicated would be $\frac{15 F . c}{2 W}$.

The solution of the two last problems might be derived from the supposition, that the whole mass was collected in the centre of gyration. The question would then be reduced to the simplest case of § 211. where the system consists only of two bodies. The preceding solution is, however, more direct.

218. If in the two preceding cases, W and F be both supposed to be sustained by the axis when the whole is at rest; the part of the weight of F of which the axis is relieved when the motion begins, is to the whole weight of F, as the force by which it is accelerated to the force of gravity.

Thus, in the instance of the sphere, the pressure on the axis is diminished by the quantity

$$\frac{F}{F + \frac{2}{15}W} \times F.$$

If W = 300 lb., F = 10 lb.; $\frac{F}{F + \frac{2}{15}W} = \frac{1}{5}$, so that the diminution of pressure is 2 lb., and therefore the whole pressure, when the motion takes place, is 308 lb.

If it were required to find directly the part of F pressing on the axis, it will come out

$= \frac{W}{W + \frac{2}{15}F} \times F$, which, in the last example, would amount to 8 lb.

Atwood on *Rectilinear and Rotatory Motion*, § 6. p. 183., &c. Frisii *Mechanica*, *Opera*, tom. II.

p. 133., &c. Vince *on the Principles of Progressive and Rotatory Motion. Phil. Trans.* 1780, p. 546., &c.

Rotation on a Moveable Axis.

219. When the impulse communicated to a body is in a line passing through its centre of gravity, all the points of the body move forward with the same velocity, and in lines parallel to the direction of the impulse communicated. But when the direction of the impulse communicated does not pass through the centre of gravity, the body acquires a rotation on an axis, and also a progressive motion, by which its centre of gravity is carried forward in the same straight line, and with the same velocity, as if the direction of the impulse had passed through the centre of gravity.

The progressive and rotatory motion are independent of one another, each being the same as if the other had no existence.

This is a consequence of the general law, That the quantity of motion estimated in a given direction is not affected by the action of the bodies on one another. The revolution of a body on its axis arises from an action of this kind, and can neither increase nor diminish the progressive motion of the whole mass.

The progressive motion, if the moving force and the body moved are given, is determined by the principles of Dynamics; and the rotatory motion is computed, as if it were about a fixed axis, by the propositions just laid down.

When one impulse only is communicated to the body, the axis on which it begins to revolve is a line drawn through its centre of gravity, and perpendicular to the plane that passes through that centre, and the direction of the impulse.

See Frisius *de Rotatione circum Axem Motum, theor.* v. *Opera*, tom. II. p. 157.

220. When a body revolves on an axis, and a force is impressed, tending to make it revolve on another, it will revolve on neither, but on a line in the same plane with them, dividing the angle which they contain, so that the sines of the parts are in the inverse ratio of the angular velocities with which the body would have revolved about the said axes, separately.

This proposition was discovered by Frisi, and is demonstrated in his works, *ubi supra*, p. 134. See also the *Cosmographia* of the same author, vol. II. p. 24.

221. If a force be impressed on any body, by which it is made to revolve on an axis, the quantity of its momentum, estimated by collecting into one sum all the products of the particles into

their velocities, and their distances from the axis of motion, is equal to the momentum of the force impressed, estimated in the same manner.

Hence, though, in consequence of the rotation of a body on its axis, a change may take place in its figure, or in its internal structure, the total quantity of its momentum will continue the same.

222. A body may begin to revolve on any line as an axis, that passes through its centre of gravity, but it will not continue to revolve permanently about that axis, unless the opposite centrifugal forces exactly balance one another.

A homogeneous sphere may revolve permanently on any diameter, because the opposite parts of the solid, being in every direction equal and similar, the opposite centrifugal forces must be equal; so that no force tends to change the position of the axes.

A homogeneous cylinder may revolve permanently about the line which is its geometric axis. It may also revolve permanently about any line that bisects that axis at right angles; but it can revolve permanently about no other line, as the centrifugal forces cannot be equal. The same is true with respect to all solids of revolution.

223. In every body, however irregular, there are three axes of permanent rotation, at right

angles to one another, on any one of which, when the body revolves, the opposite centrifugal forces exactly balance one another. These are called the Principal Axes of Rotation.

This singular theorem was first proposed by SEGNER in 1755, and first demonstrated by ALBERT EULER, in a *Mémoire* crowned at Paris in 1760. LEON. EULER has also given a demonstration of it in his treatise *de Motu Corporum rigidorum, prob.* 27. See also FRISIUS, *de Rotatione Corporum quorumcunique, theorema* VII. *ubi supra*, p. 194.

These three axes have also this remarkable property, that the momentum of inertia with respect to any of them, is either a maximum or a minimum; that is, is either greater or less than if the body revolved about any other axis. LEON. EULER, *ubi supra*. The momentum of inertia has already been defined, to be the product of every particle into the square of its distance from the axis of rotation.

SECT.

APPENDIX.

CONSTRUCTION OF ARCHES.

224. If ABCD, DCEF, EFGH, &c. (Fig. 15.) be a series of truncated wedges, resting on two immoveable supports, and having the planes of the faces that press against one another perpendicular to a vertical plane, represented here by the plane of the paper, they will be *in equilibrio*, if their weights be proportional to the differences of the tangents of the angles which these planes make with a vertical plane given in position.

Thus if the weights be as the differences of the tangents which AB, DC, EF, &c. make with the vertical MN, the whole will remain *in equilibrio*. This follows from the properties of the wedge already enumerated. Prony, *Architecture*, *Hydraul.* tom. i. § 356.

Truncated wedges disposed in this manner, form what are called *Arches* in architecture, and the most advantageous construction of them requires, that the parts should be so adjusted as to be *in equilibrio*, or to balance one another by their weight only. An arch, of which the parts balance one another in this manner, is called an *Arch of Equilibration*.

Some other definitions are necessary for understanding the construction of these arches. The truncated wedges of which the arch is composed, and which are usually of stone, are called the Voussoirs; each course of these wedges forming one voussoir. The central voussoir is called the Key-stone. The surfaces which separate the voussoirs from one another, are called the Joints. The interior curve of the arch is called the *Intrados*; the exterior, or that which limits all the voussoirs, when they are in equilibrium, is called the *Extrados*. The Abutments, are the masses of masonry, at each end, that support the arch. The beginning of the arch is called the Spring of the arch; the middle, the Crown; the parts between the spring and the crown, the Haunches. The part of the abutment from which the arch springs, is termed the Impost.

The theorem above is sufficient for the construction of arches, inasmuch as, when the curve of the intrados is given, it determines the weight of every individual voussoir. The application of it, in particular cases, admits, however, of being greatly simplified. This happens remarkably when the arch is a circle, and when the joints, (which are usually made perpendicular to the curve of the intrados), intersect in the same point.

225. In a circular arch, the weights of the voussoirs should be as the differences of the tangents of the arches, reckoned from the crown.

This is nothing more than the general proposition above, applied to a particular case. In Fig. 16. if B be the crown of a circular arch, the weight of the voussoir LKNM should be to that of the voussoir MNPO, as the tangent of BN *minus* the tangent of BK, to the tangent of BP *minus* the tangent of BN; or if, from a given point D, DH be drawn perpendicular to AC, and if the joints KL, &c. be produced, to cut it in E, F, and G, the weights of the voussoirs must be as the segments DE, EF, FG, in order to produce an equilibrium.

As the stones themselves cannot always be made in the proportion thus required, the wedges, of which they make parts, are supposed to be extended upward by courses of masonry. The whole mass included between the planes of the joints produced, as far as that masonry extends, is understood to make up the weight of the voussoir. It is the business of the theory to calculate this weight; and to construct the curve which bounds the voussoirs, when so produced.

226. In an arch, of which the intrados is a circle given in position, the depth of the key-stone being given, it is required to describe the curve of the extrados.

Let AGE (fig. 17.) be one-half of the arch, C the centre, AB the height or depth of the key-stone. From the centre E, intersect CA, with a distance equal to CK, in D; and through D, draw DH perpendicular to AC; and through G, any point in the intrados, draw GC, intersecting DH in F. Let FL be perpendicular to CE; join LB, and produce CG, till CK be equal to LB; K is a point in the extrados; and in the same way may innumerable other points be found.

The line CK is always greater than CF, but approaches to it as the arch AG increases, so that DH is an assymptote to the extrados.

The equation to the extrados is easily deduced from this construction. Draw KO perpendicular to CE. Let CO $= x$, OK $= y$, CA $= a$, CD $= b$, the equation to the curve is

$$x^2 = \frac{(a^2+b^2)y^2 - y^4}{y^2 - b^2}, \text{ and}$$

$$x = \frac{y\sqrt{a^2+b^2-y^2}}{\sqrt{y^2-b^2}}.$$

The extrados, in the case of a circular arch, is, therefore, a curve of the fourth order, very much resembling the Conchoid of NICOMEDES. It has an assymptote DH, and also a point of contrary flexure, so that it coincides very nearly with the curve in which a road is usually carried over a bridge.

Instead of b^2, it may be convenient to have its value in terms of a, and the depth of key-stone d; viz. $b^2 = 2ad + d^2$.

All this holds, whatever portion of a semicircle the arch be supposed to consist of.

227. In an elliptic arch, or one of which the intrados is a semi-ellipsis, if $2a$ be the span of the arch, or the longer axis of the ellipsis, and b the height of the arch, or the semi-conjugate axis of the ellipsis; then, if from any point in the curve, a perpendicular x be let fall on the longer axis, the weight of the key-stone is to the weight of the voussoir which has its middle at the point from which the perpendicular is let fall, as a^3 to $x^2\sqrt{a^2 + \frac{a^2-b^2}{a^2}x^2}$.

Hence, if W is the weight of the key-stone, and V that of the voussoir,

$$V = \frac{a^3}{x^2\sqrt{a^4 + \frac{a^2-b^2}{a^2}x^2}} \times W.$$

This is the same with the theorem given by Bossut (*Recherches sur l'Equilibre des Voutes*, § 14.), though the expression is somewhat changed, with a view to facilitate the numerical calculation.

228. If the weights of the voussoirs are all equal, the arch of equilibration is a catenarian curve, the same that a chain or cord of uniform thickness would assume if hanging freely, the horizontal distance of the points of suspension, being equal to the span of the arch, and the depth of the low-

est point of the chain being equal to the greatest height of the arch.

It can be easily shewn, that if the figure of the chain were reversed, the joints being such, that the force which was a pull in the first situation becomes a thrust in the second, the chain would support itself, and remain *in equilibrio.*

The equation to the catenaria, if x be the abscissa, taken from the vertex along the axis or vertical line, and y the corresponding ordinate, is

$$y = A \times \text{hyp. log.} \frac{A + x + \sqrt{2Ax + x^2}}{A}.$$

The constant quantity A may be determined by experiment; for, the chain being suspended, let a tangent be drawn to any point of the curve, and produced till it meet the axis; then, as the subtangent to the ordinate, so the length of the chain, between the given point and the vertex, to the quantity A. When A is found, the curve can be constructed.

For the properties of the catenaria, see J. BERNOUILLI, *Lectiones Calc. Integ. Lect.* 36. *Operum.* tom. III. p. 491. DAVID GREGORY, *Phil. Trans.* 1697. COTES, *Harmonia Mensurarum*, p. 108., and Notes, p. 115.

The catenaria is remarkable for this mechanical property, That a chain hanging in that curve, has its centre of gravity lower than if it were disposed in any other line, its length continuing the same, and also the points from which it is suspended.

An arch, therefore, constructed in this form, has its centre of gravity the highest possible.

229. The pressure of an arch on the piers or abutments which support it, may be estimated by considering the parts of the arch which rest immediately on the abutments to a certain height, as parts of the abutments themselves; and the remainder of the arch as a wedge, tending to separate the abutments from one another.

Thus (fig. 13.), the parts ALMS, BONT, which would remain in their places, though there were no pressure from above, may be regarded as parts of the piers; and LMDNO, the remainder of the arch, as a wedge tending to overthrow the piers by its pressure on the planes ML and ON. On these suppositions, the thickness of the piers may be determined, so that their weight shall enable them to resist this pressure. If from a point H, in the vertical EC produced, perpendiculars HK, HX, be drawn to the planes ML and ON, and if HU be taken to represent the weight of the arch MLOND, the parallelogram ID being described, HI and HD will represent the two forces which compound the force UH, and each of them will represent the force with which the arch presses on its supports ML and ON. Let RK = HI, and let RK be resolved into the two forces QK, KP, the one horizontal, and the other vertical, KQ tends to overset the pier, supposed moveable about the point F, and KP adds to its stability. If FG be bisected in W, we may suppose the weight of the pier at F acting by the lever FW, and KP by the lever FG, both tending to keep the pier in its place, while the force KQ acts by the lever KG or FY to overthrow it. Farther, though the pier is properly only FGAG′, yet it is usual to have it loaded with masonry to the height K, and even higher, so that its weight is proportional to the rectangle FK. This will give an equation between the forces, when their analytical expressions have been obtained.

Angle UHI $=\beta$, and VHI $=2\beta$, the area MLOND, $=a^2$, EC $=r$, make HU $\frac{a^2}{r}$, then IH $=$ $\frac{a^2}{2r\cos\beta}=$ KR.

Hence QK $=\frac{a^2}{2r\cos\beta\times\sin\beta}, =\frac{a^2}{r\sin 2\beta}$, and

KP $=\frac{a^2}{2r\cos\beta^2}$. Let GK $=h$, and FG $=x$, then the rectangle FK $=hx$, and the line that represents the weight of the pier will be $\frac{hx}{r}$.

Therefore,

$$\frac{hx}{r}\times\frac{x}{2}+\frac{a^2}{2r\cos\beta^2}\times x=\frac{a^2}{r\sin 2\beta}\times h,$$

and hence, $x=-\frac{a^2}{2h\overline{\cos\beta}^2}+a\sqrt{\frac{2}{\sin 2\beta}+\frac{a^2}{4h^2\overline{\cos\beta}^4}}$.

This determination of the thickness of the pier, proceeds on a hypothesis usually employed for determining the resistance of walls, &c.; but which, nevertheless, is not very conformable to fact. The hypothesis is, that the pier AF, if the weight of the arch were too great to be sustained, would fall, by turning round the point F as a *fulcrum*. Now this is not what would happen; the part of the abutment behind SM would be thrust out in the horizontal direction, till the arch had room to fall; it is therefore against the masonry immediately behind the part AM, and chiefly in a horizontal direction, that the force is exerted.

The resistance made by a wall, is, in this view of it, analogous to the friction of a body resting on a rugged surface, and has, no doubt, a given ratio to the weight of the wall, and may perhaps be regarded as equal to it. Thus the mass of building immediately behind SM, should be of force sufficient to resist the thrust KQ; or, the section of that part of the pier should be greater than $\frac{a^2}{\sin 2\beta}$.

We give this, however, only as a limit, to which it may be right to pay attention in the construction of such works.

The true principles on which the resistance of walls is to be estimated, are not sufficiently understood; and, in the mean time, a skilful and cautious engineer, can do nothing better than compute the resistance on different hypotheses, and take care that the actual strength be greater than any thing that theory points out as necessary.

230. As the wedges made by extending the voussoirs, according to the preceding theorems, would be of an inconvenient length near the spring of the arch, it will be best to substitute for them such a resistance as does not press till it is pressed upon.

Thus, if on the pier immediately behind the arch, there be raised a body of masonry formed of horizontal courses of large stones, the voussoirs being continued no farther than to come in contact with the ends of those courses, the purpose of equilibrium will be answered, supposing the work of sufficient strength, because any of the voussoirs pressing against the horizontal courses, will have its own defect of weight immediately supplied by their reaction. Each voussoir will then receive, with mathematical exactness, the precise degree of pressure required for the equilibrium, and can neither have more nor less than that quantity. The upper part of the arch is here supposed to be loaded in the manner that strict theory requires; to which there does not appear to be any practical objection. If the pier be an intermediate one between two arches, the opposite pressure on the horizontal abutting courses will render their effect more certain.

On the subject of Arches, see HUTTON's *Principles of Bridges.* BOSSUT, *Recherches sur l'Equilibre des Voutes, Mém. de l'Acad. des Sciences,* 1774 and 1776; also *Mechanique* of the same author, edit 1802, 1802, p. 383., &c. PRONY, *Architecture Hydraulique*, tom. I. § 358., &c. ATWOOD's *Treatise on the Construction and Properties of Arches*, published in 1801. This last work is particularly commendable, for the great accuracy of the investigation, the minuteness with which the subject is considered, and the various experiments brought in support of the theory. A *Supplement*, in two parts, was added to it; the last in 1804.

STRENGTH OF TIMBER.

231. If a beam of wood, of which the section is any given figure, having its area $= a^2$, have one end firmly fixed in a wall; if s be the strength of each fibre estimated laterally; d the distance of the centre of gravity of the section, from the edge where the fracture terminates, and round which the beam, when it is broken, turns as on a hinge; let l be the length of the beam, measured from the wall from which it projects, to the end at which a weight W is hung, just sufficient to break it; $W \times l = s\, a^2 \times d$, and $W = \frac{s\, a^2\, d}{l}$.

The weights, therefore, required to break different beams, are as the strength of the individual fibres, multiplied into the area of the sections, and into the distance of the centre of gravity of the sections from

from the points round which the beams turn in breaking, divided by their lengths.

We are indebted to GALILEO for this theorem. He was the first who attempted to reduce the strength of the materials used in the mechanical arts, to the measures of geometry and arithmetic. *Dialogo Secondo*, *Opere*, tom. III. p. 68., &c. See also, *ibid.* p. 213., a *Commentary* on what GALILEO has written on this subject, begun by VIVIANI, and completed by GUIDO GRANDE.

It is obvious, that the hypothesis of the beam breaking short over, and turning as on a hinge, is not the precise fact; and therefore the conclusion deduced from this hypothesis cannot be expected to be more than an approximation. It does not appear, however, that more refined views, or more complicated speculations, have led to results agreeing better with experiment than the simple hypothesis of GALILEO.

232. In a beam, of which the section is a rectangle, having the breadth b, and the depth c, $W = \frac{s\,b\,c^2}{2\,l}$. The strength, therefore, all other things remaining the same, is as the breadth multiplied into the square of the depth.

This proposition has been brought to the test of experiment by BUFFON, in a great number of trials, made with great accuracy, and on a large scale.

The beams he subjected to trial were of oak, from seven to twenty-eight feet in length, and from four inches square to eight inches.

It appears from them, that the quantity called s in the above formulas, or the strength of the ligneous fibre, is nearly proportional to the specific gravity, or to the weight of a given bulk, for example, a cubic foot of the timber. The weights required to break the beams were found to be very nearly as the quantities $b c^2$, the length remaining the same. The real strengths, however, fell a little short of the computed, and the more, the longer the beam. In beams of different lengths, the variation of strength did not follow the inverse ratio of the lengths near so exactly as it did that of the other quantities included in the formula. *Histoire Naturelle par* M. DE BUFFON; *Supplement*, tom. III. p. 255., &c. 12mo edit.

233. In a beam, of which the section is a triangle, having the base b, and the depth or perpendicular p, when the base is uppermost, $W = \frac{s b p^2}{3 l}$; but when the vertex of the triangle is uppermost, or the edge of the prism, $W = \frac{s b p^2}{6 l}$.

The strength of a prismatic beam is therefore twice as great when a face is uppermost, as when the opposite edge is uppermost.

In like manner, in a cylindric beam, of which the radius is r, $W = \frac{\pi s r^3}{l}$.

In a cylindric tube, the radius of the external surface being r, and of the internal r',

$$W = \frac{\pi s r (r^2 - r'^2)}{l}.$$

The strength of the tube is therefore to the strength of the same quantity of matter, formed into a solid cylinder of the same length, as r to $\sqrt{r^2 - r'^2}$.

A tube may, therefore, be much stronger than the same quantity of matter in a solid form. This is known to be agreeable to experience. But it is also said, that a tube of metal has been found to support a greater transverse strain than a solid cylinder, of the same diameter; or that a solid cylinder, when bored in the direction of the axis, and a considerable part taken away, was stronger than before.

This must undoubtedly arise from a change taking place in the position of the fulcrum or hinge round which the fracture is made. In the case of a cylinder, and, indeed, of all solids, the fulcrum is not the mere outward edge, but a point in the interior; on the one side of which the fibres are elongated, and on the other crushed together. The point, then, which serves as the fulcrum, will be found within the solid, at a greater or less distance, as the parts resist lengthening more than crushing. The consequence of this is, that when the centre of gravity and the fulcrum are brought nearer to one another, the strength of the beam or bar is diminished. When the heart of a solid mass is cut out, as is supposed of the cylinder, the fulcrum, or the axis of the fracture, is perhaps kept nearer to the surface than when the whole is a solid mass. This, at least, seems to be the most probable account that can, at present, be given of a phenomenon not a little paradoxical, and not yet sufficiently examined.

234. In similar beams, or solids, of the same substance, the strength increases as the square of the lengths; but the weight increases as the cube. There is of course a limit, which, if a beam of a given shape, and of given materials, were to reach, it could only bear its own weight, and would be incapable of farther increase.

EMERSON'S *Mechanics*, scholium at the end of section VIII. p. 111.

235. When a beam, instead of projecting from a wall, is supported at both ends, it must break in the middle, and the termination of the fracture will be on the upper side; and hence, when a rectangular beam is supported at both ends, it is able to carry twice as great a load as when it is supported only at one end.

When the section of the beam is such, that the centre of gravity does not coincide with the centre of magnitude, as the termination of the fracture, by being brought from the under to the upper side, must change its distance from the centre of gravity of the section, a change of strength will take place also on that account.

236. When a beam is supported at both ends, the weight which it is able to bear at any point, is inversely as the rectangle under the segments into which it is divided by that point.

A beam, therefore, supported at both ends, and of the same section throughout, is weakest in the middle.

A beam supported at both ends, and of a given breadth, will be every where of the same strength, if its longitudinal and vertical section be an ellipse, having the beam itself for its greater axis.

237. From a given cylinder, to cut out the strongest rectangular beam possible.

Let the circle ACB (fig. 12.) be the base of the cylinder; at the extremity A, of the diameter AB, draw the tangent AH equal to the cord of 90°, draw HK to the centre K, and let it cut the circumference in C. If CE be drawn parallel to AB, and CD at right angles to it, the rectangle DE, under these lines, is the section of the strongest rectangular beam that can be cut out of the given cylinder.

For it is easily shewn, that $CD^2 \times CF$, to which the strength of the beam is proportional (§ 232.), is greater in the rectangle thus determined, than in any other that can be inscribed in the circle.

As from the construction $CL^2 = 2\,LK^2$, if the radius be called r, $LK = \frac{r}{\sqrt{3}}$, and $CL = r\sqrt{\frac{2}{3}}$; so $CF = \frac{2r}{\sqrt{3}}$, and $CD = 2r\sqrt{\frac{2}{3}}$. CD is, therefore, to CF, nearly as 7 to 5.

In practice, as the tree from which the rectangular beam is to be cut, can hardly ever be truly cylindric, the first thing to be done is, to inscribe, by trial, the greatest circle possible, in the section of the tree. The rectangle may then be found by the construction above.

238. If it is required to find the strongest rectangular beam under a given perimeter or circumference, divide the given circumference into six equal parts, and take two of these for the depth of the beam, and one for its breadth.

These propositions are understood to be applicable, not only to beams of wood, but to bars of iron, or any other metal.

239. If a beam, having one end firmly fixed in a wall, project from it with a certain inclination to the horizontal plane, the weight which it will support at its extremity, is greater than that which it would support if horizontal, in the ratio of the square of the radius to the square of the cosine of the inclination.

That is, if W be the weight which a beam can support in a horizontal position, $\frac{W}{\cos i^2}$ is the weight it can support when inclined to the horizon at the angle i.

The proof of this proposition proceeds on two principles, that the momentum of the weight is diminished in the oblique beam, by its perpendicular distance from the fulcrum being diminished, and that the resistance of the beam is increased by the centre of gravity of the section being carried farther from the fulcrum. Each of these being in the inverse ratio of the cosine of the inclination, their joint effect is in the inverse ratio of its square.

It is implied in this reasoning, that the resistance of each fibre is the same to the oblique and to the direct fracture. Though this seems probable, it is not altogether certain, and it would be useful to have experiments directed to the clearing up of this difficulty. If the individual fibres resist fracture also in the inverse ratio of the cosine of the obliquity, the strength of the beam will be in the inverse ratio of the cube of that cosine. This is observed by GUIDO GRANDE, in the treatise already referred to. *Opere di* GALILEO, tom. III. p. 271.

240. The theory in the preceding proposition being admitted, the force of a beam to resist the action of any strain, will be inversely as the square

of the sine of the angle which the direction of that strain makes with the direction of the longitudinal fibres of the wood.

According to this, the strength of the beam to resist a force applied to it endwise is infinite. This is no doubt false; but the force in the case of a longitudinal pull or thrust is so great, that no practical error will arise from it. The cause of the error in this extreme case is, that the beam does not, nor cannot, obey the law of *continuity*, which would require, that when, by varying its inclination, it becomes quite upright, its longitudinal section should be infinitely extended.

From this theorem, it is evident, that great advantage is obtained by making the strain on all timber as oblique or as nearly longitudinal as possible. This is the great principle in the construction of roofs and centres of bridges. The principle of the *Arch* should also be combined with it, so that pressure downwards may be so applied as to produce a pull or thrust upward. See the article by Professor Robison *on the Strength of Timber*, in the *Encyclopædia Britannica*.

In the preceding theorems, no account is made of the weight of the beams, a circumstance that it is often necessary to be considered. This may always be done by supposing the weight of the beam to be an addition to its load, collected in its centre of gravity. On this the following theorem is founded.

241. If a rectangular beam, fixed in a wall, project horizontally to the length l; if the depth of its section be a, and the weight of a cubic foot of the timber be p; then if $l = \sqrt{\frac{s a}{p}}$, the beam will be just able to support its own weight.

To apply this theorem, the value of s must be found from experiment. In the case of a rectangular beam, we have $s a b . \frac{a}{2} = W \times \frac{l}{2}$, supposing the beam supported at both ends, § 235.; and therefore $s = \frac{W \times l}{a^2 b}$. In BUFFON's experiments, $a b$ was a square; and in one where a was six inches, and l equal seven feet, W was 19250 lb. Therefore, $s = \frac{19250 \times 7}{\frac{1}{8}} = 19250 \times 56 = 1078000$. In this same experiment, the weight of the beam was 128 lb., and therefore a cubic foot weighed 73.2 $= p$, so that $s = \sqrt{\frac{s a}{p}} = 85.8$ feet nearly.

The following Table contains the mean results of BUFFON's experiments. They were made on oak beams, cut from trees newly felled, and full of sap. The weight of the timber varied from 73 to 75 French pounds to the cubic foot; its strength varied in the same proportion.

TABLE.

Length of the Beams.	Breadth or Thickness.			
	4 inches.	5 inches.	6 inches.	7 inches.
Feet.	Lb.	Lb.	Lb.	Lb.
7	5312	11525	18950	
8	4550	9787½	15525	26050
9	4025	8308¼	13150	22350
10	3612	7125	11250	19475
12	2987	6075	9100	16175
14		5300	7475	13225
16		4350	6362½	11000
18		3700	5562½	9245
		3225	4950	8375

Buffon, *Hist. Naturelle, Supplement*, tom. III. p. 260. 12mo edit. Paris 1779. For a comparison of the strength of different kinds of wood, &c. See **Emerson's** *Mechanics, section* VIII. *scholium* at the end. **Muschenbroek**, § 1127, on the *Strength of Timber*, and § 1129, &c. on the *Strength of Metals*. See also **Coulomb**, *Application des Regles de* Maximis *et* Minimis *à quelques Problémes relatifs à Architecture, Mémoires Presentés*, vol. VII. (1773), p. 343., &c.

HYDROSTATICS.

SECT. I.

PRESSURE OF FLUIDS.

242. A FLUID is a body so constituted, that its parts are ready to yield to the action of any pressure, however small, in any direction.

NEWTON, *Principia*, lib. II. sectio 5. *in initio*. ARCHIMEDES *de Insidentibus Humido.*

The science which applies the principles of Dynamics to fluid bodies is Hydrodynamics, and is divided into four parts, according as fluids are incompressible or elastic, and according as their equilibrium or their motion is considered.

When fluids are incompressible, like water, the science which explains the laws of their equilibrium is called *Hydrostatics*, and that which explains the laws of their motion is called *Hydraulics*. When they are compressible and elastic, like air, the science which treats of the laws of their equilibrium is called *Aërostatics*, and that which treats of their motion is *Pneumatics*.

The fluids here treated of, are supposed to gravitate in lines either parallel, or directed to a centre, or to several centres.

243. The surface of every fluid, when at rest, is horizontal, or perpendicular to the direction of gravity.

If the directions of gravity are all parallel, the surface of the fluid is a plane: If they converge to a point, the surface of the fluid is a portion of a spherical surface, having that point for a centre.

BOSSUT, *Hydrod.* tom. I. § 19. CAVALLO, vol. II. chap. ii. prop. 2.

Levelling is the art of drawing a line at the surface of the earth, to cut the directions of gravity every where at right angles.

This line is nearly an arch of a circle, and if L be the length of any such arch in English miles, and D the depression, in feet, of one extremity of it, below a tangent drawn to it at the other extremity, $D = \frac{2L^2}{3}$.

If a communication, by means of a tube or pipe, either straight or crooked, be made between the water

ter in one vessel, and that in another, the surfaces of both will come to be at the same level before the water is at rest; or if there is not water sufficient to bring both to a level, the whole will be accumulated in the lowest.

If water is permitted to flow freely from one vessel to another, it will never be at rest till it is the lowest possible.

244. The fluid contained in any vessel, being at rest, and subjected to the sole action of gravity, any particle of it is pressed in all directions, (vertically, horizontally, and obliquely), by the same force, viz. the weight of the column of water perpendicularly incumbent on it.

Bossut, *Hydrodynamique*, vol. I. § 23.

245. If the fluid contained in any vessel be at rest, and subjected to the action of gravity only, the pressure on an indefinitely small area, at any point of the bottom or sides, is perpendicular to the plane of that area, and equal to the weight of a vertical column of the fluid, standing on it as a base, and reaching to the surface.

Hence the pressure on any part in the bottom or sides of a vessel, depends entirely on the depth of the fluid at that point, and not at all on the extent of the fluid in a horizontal direction.

246. The same being supposed, as in the last proposition, the whole pressure sustained by any portion whatever of the bottom or sides of the vessel, is equal to the weight of a column of the fluid having for its base the surface pressed on (extended into a plane if necessary), and for its altitude the mean depth of the incumbent fluid.

COTE's *Hydrostatical Lectures*. BOSSUT, *Hydrodyn.* vol. I. § 26.

The mean depth of any portion of the surface of the bottom or sides of the vessel, is the same with the distance of the centre of gravity of that portion below the surface of the fluid.

The strength of the walls necessary to resist the pressure of a fluid of any given depth, is determined by this proposition.

If the figure of the vessel be such as is represented, (fig. 20,) the pressure on every point of the bottom is the same as if it were filled with fluid every where to the height FEG. Hence the pressure of a fluid on the bottom of a vessel may be very great, while the weight of the fluid is very small; and hence also the addition of a small quantity of water in the neck or tube HE, may increase the pressure on CD in a very great proportion. This is called the *Hydrostatic Paradox*.

247. A body immersed into a fluid is pressed upwards by a force equal to the weight of the fluid it displaces, and the direction of that force

passes through the centre of gravity of the part immersed.

The proposition holds, whether a body sink in a fluid or float on its surface.

When a body floats, the weight of the water displaced is equal to the weight of the body.

248. The difference between the absolute weight of a body, and its weight when entirely immersed in a fluid, is the same with the weight of a quantity of the fluid equal in bulk to the body.

If W be the weight of a body *in vacuo*, (which we may suppose to be nearly the same with its weight in air), and if W′ be its weight in water, W — W′ is the weight of a quantity of water equal in bulk to the body:

It follows from § 30, that the weight of any body divided by the weight of an equal bulk of water, measures the *specific gravity* of the body. The specific gravity of a body weighed as above, is

$$\frac{W}{W - W'}.$$

Hence the specific gravities of bodies are found by weighing them, first in air, and then in water. The instrument with which this experiment is made is called the *Hydrostatic Balance.*

249. Let it be required, by the hydrostatic balance to determine the specific gravity of a body so light as not to sink in water.

Weigh

Weigh the body first in air, and then by a slender thread attach to it a heavier body, that the two together may sink; the heavier body being previously weighed, first in air, and then in water. Let the compound body thus formed be weighed in water; then from the weight lost by it, subtract the weight lost by the heavier body, when weighed singly; the remainder is the weight lost by the lighter body, by which if its weight be divided, its specific gravity will be found.

Cavallo, vol. ii. p. 61. His solution is a little different from this, but the result is the same. See the example, *ibid.*

250. If the same body be weighed in two different fluids, the weights lost will be to one another as the specific gravities of the fluids.

By help of this proposition, the specific gravities of different fluids may be compared together. The specific gravities of different fluids may also be found, by weighing equal bulks of them.

If a body float on different fluids, the bulks of the part immersed will be inversely as the specific gravities of the fluids. It is on this principle that the hydrometer, aërometer, &c. are constructed.

See *Description of a Hydrometer*, Cavallo, vol. ii. p. 66.

251. If the specific gravity of air be called m, that of water being 1, if W be the weight of any

body in air, and W' its weight in water, then $W+m(W-W')$ is its weight *in vacuo* very nearly.

In the mean state of the atmosphere, $m=.00122$ nearly.

252. If s be the specific gravity of a body, ascertained by weighing it in air and water, and m be the specific gravity of the air at the time when the experiment was made; the correct specific gravity, or that which would have been found if the body had been weighed in a vacuum instead of air, is $s+m(1-s)$.

When the body is heavier than water, this correction is subtractive; when lighter, it is additive.

The manner of finding the specific gravity of air will be afterwards explained.

253. If B be the bulk of any body in cubic inches English, W its weight in grains, and S its specific gravity, that of water being 1;

$$B=\frac{W}{(252.576)\,S}.$$

This theorem is founded on an experiment of Sir G. Shuckborough's, (*Phil. Trans.* 1798), by which the weight of a cubic inch of distilled water, of the temperature 66°, was found to be 252.422 grains, which, reduced to the temperature 60°, is 252.576. Cavallo, vol. II. p. 98.

The logarithm of 252.576 is 2.4023921.

If W is expressed in lbs. Troy, it must be multiplied by 5760; if in lbs. avoirdupois, by 7002. If B is required in cubic feet, its value must be divided by 1728.

SECT. II.

SOLID BODIES FLOATING ON FLUIDS.

254. A solid floating on a fluid specifically heavier than itself, will be *in equilibrio* when it has sunk so far, that the weight of the fluid displaced is equal to the weight of the whole solid, and when the centres of gravity of the whole solid, and of the part immersed, are in the same vertical line.

ARCHIMEDES, *de Humido Insidentibus*, lib. I. BOSSUT *Hydrodyn.* tom. I. chap. 12.

Hence every solid of revolution, and in general every solid which has an axis that divides it into two parts equal and similar, if it be specifically lighter than a fluid, and if it be placed in the fluid with its axis vertical, may sink to a position in which it will remain *in equilibrio*.

There are always two opposite positions of equilibrium in such bodies; but there is only one of them in which the body can float permanently.

255. A prism, of which the triangle ABC (fig. 21.), is a transverse section, being supposed homogeneous, and its specific gravity being less than that of a fluid in which it is immersed, but having a given ratio to it, required the different positions in which it will be in equilibrium.

Let the prism float with the angle ABC downward, and let DE be the common section of the plane of the triangle with the surface of the fluid, or what is called the *water line*. The area of the triangle BDE is given, being to that of the triangle ABC, as the specific gravity of the prism to the specific gravity of the fluid; therefore the line DE touches a given hyperbola, described with the assymptotes AB, BC, and is bisected in the point H where it touches that curve. Bisect AC in F, draw BF, BH; take $BG = \frac{2}{3} BF$, and $BK = \frac{2}{3} BH$, G is the centre of gravity of the triangle ABC, and K that of the triangle DBE, and therefore, because of the equilibrium, GK is a vertical line, as also FH, which is parallel to it. Therefore FH is perpendicular to DE, and consequently to the hyperbola which DE touches in H. The position of FH, since the point F is given, may therefore be found; and as many perpendiculars as can be drawn from that point to the hyperbola, so many positions are there in which the prism may float with the angle ABC downward.

The different positions of FH are determined, by the roots of a biquadratic equation. Draw FE and HL perpendicular to BC; let $BE = a$, $EF = b$, $BL = x$, BL$=x$, LH$=y$, and the constant rectangle HL×LB$=c^2$; then FH$^2=(a-x)^2+(b-y)^2$ and $xy=c^2$; whence by the method of *maxima* and *minima*, we obtain $x^4-ax^3+bc^2x=c^4$, from which the different values of BL may be determined.

A geometric construction may be obtained, by substituting xy for c^2, whence we get

$$x^2-ax=y^2-by,$$

an equation to a hyperbola, the intersection of which, with the given hyperbola, will determine the different values of BL, and consequently the different positions of FH.

See the solution of other problems of this kind, ARCHIMEDES *de Humido Insidentibus*, lib. II. BOSSUT, *Hydrod.* tom. I. chap. xii. § 166. &c.

256. When the equilibrium of a floating body is disturbed, or when the centres of gravity of the whole and of the part immersed are not in the same vertical line, if a vertical plane be made to pass through both those centres, the body will revolve on an axis passing through its centre of gravity, and perpendicular to that plane.

This follows from what has been proved concerning the rotation of bodies, § 219.

BOSSUT, *Hydrod.* tom. II. § 179. ATTWOOD, *Phil. Trans.* 1796.

257. A body may be so constituted, that in every position the centres of gravity of the whole, and of the part immersed are in the same vertical line, and in such a body the equilibrium cannot be disturbed by any force that only tends to make it revolve on an axis.

> **A homogeneous sphere is a body of this kind, and so also is a cylinder floating with its axis horizontal. These have no tendency to maintain one position more than another, and their equilibrium is called the *equilibrium of indifference.***

258. Some floating bodies, when their equilibrium is disturbed, return to their position after a few oscillations backwards and forwards. Others, when their equilibrium is ever so little disturbed, do not resume their former position, but revolve on their centres of gravity, till they come into another position, when they are again *in equilibrio.*

> **The equilibrium in the first case is said to be stable, in the latter case to be unstable. In this last case, the bodies are said to *overset.***

259. When a floating body is made to revolve from the position of equilibrium, if the line of support (that is, the vertical, through the centre of gravity of the immersed part) move so as to be on the same side of the line of pressure (or the

vertical through the centre of gravity of the whole) with the depressed part, the equilibrium is stable, and the body will resume its former position.

260. The same things being supposed, if the line of support is on the same side of the line of pressure with the elevated part, the equilibrium is unstable, and the body will overset.

Bossut, *Hydrod.* tom. i.

261. If a body float on the surface of a fluid, the force tending to make it revolve about its centre of gravity is equal to the weight of the body, acting by a lever, the length of which is the horizontal distance between the line of pressure and the line of support.

Attwood, *Phil. Trans.* 1796, p. 61.

When this distance is nothing, or when the two centres are in the same vertical line, the force tending to make the body revolve, is equal to 0, as already stated, § 257.

When the body is any how inclined from the state of equilibrium, and when the line of support is on the same side of the centre of gravity with the depressed part, the lever by which the weight acts is accounted affirmative, and the force tends to establish the equilibrium.

When the line of support is on the same side of the line of pressure with the elevated part, the force is accounted negative, and tends to overset the body.

262. If in a floating body, of which the transverse section is the same from one end of the body to the other, a be the length of the water line, c^2 the area of the section of the immersed part, d the distance between the centre of gravity of the whole and the centre of gravity of the immersed part, and i an indefinitely small inclination from the position of equilibrium; the momentum of the force tending to restore the equilibrium is $\left(\frac{a^3}{12c^2} - d\right) W \sin i$.

If $\frac{a^3}{12c^2} > d$, the force tends to restore the body to its state of equilibrium.

If $\frac{a^3}{12c^2} = d$, there is no force tending either to restore or destroy the equilibrium.

If $\frac{a^3}{12c^2} < d$, the force becomes negative, and tends to overset the body.

When W remains the same, the *stability* is proportional to $\left(\frac{a^3}{12c^2} - d\right) \sin i$.

When the centre of gravity of the body is lower than the centre of gravity of the immersed part, d is negative, and the quantity $\frac{a^3}{12c^2} - d$ is affirmative, whatever be the magnitude of $\frac{a^3}{12c^2}$.

If in the axis of the solid, or in the line passing through the two centres, there be taken a point distant from the centre of the immersed part, by $= \frac{a^3}{12c^2}$, this point is called the *metacentre*, because the centre of gravity must be lower than it, in order that the body may float with stability.

By the centre of gravity of the immersed part, is always understood its centre of gravity, supposing it homogeneous. It is in fact the centre of gravity of the water displaced, and were perhaps better distinguished by the name of the centre of *buoyancy*.

263. When a floating body revolves about a given axis, the positions of equilibrium through which it passes, are alternately those of stability and instability.

For between a state in which a body has a tendency to remain, and another in which it has also a tendency to remain, as these tendencies are opposite to one another, there must be an intermediate position,

tion, in which the tendency to remain is equal to nothing.

Atwood, *Phil. Trans.* 1796, p. 63.

264. If a rectangular parallelopiped float in a fluid, with its altitude a perpendicular to the surface; if its breadth be b, and its specific gravity n, that of the fluid being 1, its stability will be as $\frac{b^2}{6} - n(1-n)a^2$.

When it has no stability, $\frac{b^2}{6} - n(1-n)a^2 = 0$, and

$$a = \frac{b}{\sqrt{6n(-n)}}.$$

If $n = \frac{1}{2}$, as is nearly the case with fir,

$$a = \frac{b}{\sqrt{3 \times \frac{1}{2}}} = b\sqrt{\frac{2}{3}} = \frac{5b}{6}$$ nearly. The truth of this conclusion may be shewn by experiment.

Atwood, *ubi supra*, p. 67. Young's *Analysis of the Principles of Natural Philosophy*, p. 212. § 15.

265. As the force impelling the floating body in its vibrations, is proportional to the sine of the in-

clination of its axis, or the angle which it makes with the vertical, and as this is the same law that regulates the vibrations of a pendulum in a small arch of a circle, a pendulum may be found, the vibrations of which are isochronous with the small vibrations of a floating body.

ATWOOD, *Phil. Trans. ubi supra*, p. 123. BOUGUER, *Manœuvre des Vaisseaux*, lib. I. § iii. chap. 7.

On the subject of this section, see also EULER'S *Theory of the Construction and Properties of Vessels*, translated by H. WATSON, chap. 4, 5 & 6., &c. GREGORY'S *Mechanics*, vol. I. p.

SECT. III.

PHENOMENA OF CAPILLARY TUBES.

266. IF a capillary tube of glass, that is, a tube of which the bore is less than one-tenth of an inch, be immersed in water, the water will rise within the tube to a greater height than it stands at on the outside; and this height is nearly in the inverse ratio of the diameter of the tube.

For a tube of which the bore is $\frac{1}{100}$th of an inch, the rise is 5.3 inches. So the product of the diameter of the tube, into the height to which the water rises,

ses, is a given quantity, and nearly = .053, in parts of a square inch.

It follows also from the above, that the weights of the columns of water sustained in capillary tubes, are as the diameters of these tubes.

Though the rise of water above its natural level is most manifest in small tubes, it appears, in a degree, in all vessels whatsoever, by a ring of water formed round the sides, with a concavity upwards.

267. Capillary suspension takes place, though the tube be not immersed in water, providing a drop of water adhere to the lower end of it.

268. The surface of the water in a capillary tube is concave upwards; and if by taking the tube out of the water, and inclining it, the fluid be made to move along it, the concavity appears at both ends of the column, and is the same in figure and size whether the tube be held perpendicularly, horizontally, or obliquely.

269. When by inclining the tube, the column of water is made to move, it appears to suffer resistance as it approaches either end, and does not completely reach the end, till the tube is either altogether, or very nearly, inverted.

When the tube is placed in the water, however far it be thrust down, the water within never reaches the top, till the tube is completely immersed.

270. If a capillary tube composed of two cylinders of different bores, be immersed in water, first with the widest part downwards, and afterwards with the narrowest, the water will rise in both cases to the same height.

If the smaller end is such as to require the whole to be filled by suction, the water stands at the same height as if the whole tube were of the bore of the upper part. This experiment, however, does not succeed *in vacuo*, and therefore the water in the wide part of the tube, must be considered as sustained by the pressure of the air.

271. If two plates of glass be kept parallel, and near to one another, and if their ends be immersed in water, the water will ascend between them to half the height it would rise to in a tube having its diameter equal to the distance of the plates.

Newton's *Optics*, Book iii. query 31. Haüy, *Traité Elementaire de Physique*, tom. i. § 333. edit. 1806.

When the plates make an angle with one another, if they be immersed with the line of their intersection vertical, the water will ascend between them, and form a hyperbola.

La Place caused experiments to be made with one tube placed within another, so that their axes coincided, and found that the water ascended in the space between them only half as high as it would have done in a single tube in which the diameter of the bore was equal to the distance of the two tubes from one another. **Haüy**, *ibid.*

272. If a glass tube of a small bore be immersed in mercury, the mercury does not rise within the tube to the height at which it stands on the outside: its surface within is *convex*, as it is also without, all round the tube.

273. If into a conical capillary tube, held in a horizontal position, a drop of water be introduced, it will run toward the narrow end; but if a drop of mercury be introduced, it will run toward the wide end.

La Place, *Mechanique Celeste*, Lib. x. Supplement, p. 6.

274. Whether a fluid rise or fall between two vertical and parallel planes, immersed in the fluid at their lower extremities, the planes tend to approach one another.

It is from this tendency, that two small vessels of glass, of a parallelepiped form, floating on water or

mercury, unite, whenever they approach near to one another. La Place, *ibid.* § 11.

The phenomena here enumerated, clearly prove the existence of an attractive force between water and glass, but require to be carefully compared before they inform us of the manner in which that force is exerted. The fact that points most directly to the place where the force resides, is that of the concavity of the surface, mentioned at § 268.

275. From this fact it may be inferred, that a narrow ring or zone of glass immediately above the surface of the water, exerts its force on the water, and it is found that the phenomena of capillary tubes, may be explained by supposing, that the attraction of the glass, combined with the weight of the water, and the cohesion of its particles, is the cause of the concavity just mentioned, or of the little *meniscus* of water, MIOKN (fig. 22.), which terminates the column. This meniscus is, therefore, a body of water stretched across the tube, and sustained there by the attraction of the glass, while it exerts its own attraction on the particles of the column immediately underneath, by which means the gravity of those particles is diminished, and the water rises in the tube above its level on the outside, to supply their deficiency of weight.

This is the theory of LA PLACE. He has determined the superficies of the meniscus to be nearly spherical, and its attraction to be equal to that of a *spherule* of water of the same diameter with itself. The ascent of the water, in the inverse ratio of the diameter of the capillary tube, is thus accounted for; the attraction of the meniscus being directly as its diameter, or as the diameter of the tube, that is, as the weight of the water sustained.

It is understood in all this, that the attraction of the particles reaches beyond the nearest, but not to any sensible distance.

In the case of mercury, glass either repels the fluid, or attracts it less than its particles attract one another. Hence the convexity of the surface that terminates the column, and the depression of that surface below the natural level.

The author of this theory considers the concave surface of the fluid, and the attraction which the meniscus exerts, as the principal cause of the phenomena of capillary tubes. *Ibid.* p. 5. We confess, that we cannot see the matter precisely in this light. The principal and primary cause must certainly be, that attraction which sustains the meniscus, and enables it to act on the water below without being drawn out of its place. It is not the concavity of its surface that makes the water in the tube press less on the bottom than if its surface were plain, but it is the attraction of the glass that produces, in a manner equally direct, both the concavity and the diminution of the pressure.

276. The reason why the water between the two narrow plates of glass rises only to half the height it does in a capillary tube, of a diameter equal to the distance of the plates, is, that the quantity of the attracting zone is nearly halved.

The same holds of the tubes, inserted the one within the other. Any small column of the water between them, is supported by the attraction of the glass only on two sides.

So also, in the movement of the drop to the small end of a conical capillary tube; if the attraction of a particle of the tube be resolved into two, one in the direction of the axis of the tube, and the other at right angles to it, the former will be directed towards the vertex of the cone. If the force is repulsive, the part in the direction of the axis will be toward the base. On the same principle may be explained the tendency of two plates to unite, whether water rises or mercury sinks between them.

The theory of La Place, so far as the attraction of the tube is concerned, has a great affinity to that of Dr Jurin, *Phil. Trans.* NO. 363. art. 2. In what respects the form of the upper surface, it is founded on the same principles with Clairaut's explanation, *Theorie de la Figure de la Terre, tirée des Principes de l'Hydrostatique*, § 59.

277. Clairaut has proved, that if the law by which the matter of the tube attracts the fluid, be the same with that by which the parts of the fluid attract one another, the fluid will rise above the level whenever the intensity of the first of these attractions exceeds half the intensity of the second. If it is exactly the half, the surface of the column within the tube will be a plane, on a level with the surface without.

Theorie, &c. § 60.

In some cases, the attraction of the bottom has an evident effect; when the lower end of the capillary tube is stopped by the finger, and the tube taken out of the water, if the finger is withdrawn, a drop forms, and the water stands higher in the tube than when it was immersed in the water. La Place, *ibid.* § 15. This seems to arise from the action of the bottom and outside of the tube on the drop, by which the column of water in the tube is lifted up, as it were, and raised to a higher level.

The circumstance that has so long rendered the theory of capillary tubes a matter of doubt, is the difficulty of finding a perfect *experimentum crucis*, or a fact which can be explained by one theory only. It cannot be said, that even the theory of CLAIRAUT and LA PLACE possesses the demonstrative evidence which such an experiment would afford.

HYDRAULICS.

HYDRAULICS.

SECT. I.

FLUIDS ISSUING THROUGH APERTURES IN THE BOTTOM OR SIDES OF VESSSELS.

278. IF two bodies be uniformly accelerated over distances that are inversely as the accelerating forces, they will acquire equal velocities.

This proposition is given as a lemma to that which follows.

279. The velocity with which a fluid issues from an infinitely small orifice in the bottom or side of a vessel that is preserved full, is equal to that which a heavy body would acquire by falling from the level of the surface to the level of the orifice.

BOSSUT, *Hydrod.* tom. I. § 218.

Therefore, let d be the depth of the fluid; g the velocity acquired by a falling body in one second; v the velocity with which the water issues;

$v = \sqrt{2gd}$.

If the velocity with which the fluid issues from the orifice, were turned directly upward, it would carry it to the level of the surface of the fluid.

280. The quantity of water that issues in one second, through a given orifice, is equal to a column of water having the area of the orifice for its base, and the velocity with which the fluid issues for its altitude.

Hence, if a^2 be the area of the orifice, and d the depth of the water above the orifice, and q the quantity of fluid running out in one second,

$q = a^2 \sqrt{2gd}$.

Of the quantities of q, a and d, any two being given, the third may be found:

$$a^2 = \frac{q}{\sqrt{2gd}}; \quad d = \frac{q^2}{2ga^4}.$$

The quantity of water that issues through an orifice, is as the area of the orifice multiplied into the square-root of the depth.

When the experiment is made, it is found that A must not be taken equal to the exact area of the surface. The vein of water that issues through the small orifice, suffers a contraction, by which its section has been found to be diminished in the ratio nearly of 5 to 7. Therefore we must take $\frac{5}{7}$ of the above result for the true value of g, which is, therefore, $\frac{5}{7} a^2 \sqrt{2gd}$. As $\frac{5}{7}$ is near $= \frac{1}{\sqrt{2}}$, we may, therefore, make $Q = a^2 \sqrt{gd}$.

As $\sqrt{gd}$ is the velocity acquired by falling through $\frac{d}{2}$, it has been supposed by some writers, that the water issues only with the velocity that would be acquired by falling through half the depth of the fluid. See CAVALLO, vol. II. chap. 7. p. 168.

281. Water issuing through an oblique pipe in the side of a vessel which is kept full, describes a parabola in a vertical plane, passing through the axis of the pipe; and the directrix of the parabola is the line of common section of the said vertical plane with the surface of the water produced.

That the curve is a parabola, follows from the theory of Projectiles, § 89.; and that the directrix is the same, whatever be the angle of elevation, follows from this, that the water issues with the same velocity whatever direction the axis of the pipe has, viz. that which corresponds to the height of the surface of the fluid above the orifice.

282. The time that a cylindric vessel, (the area of its base being b^2), requires to empty itself by a hole in the bottom, of which the area is a^2, the depth at the beginning of this discharge being d, is $\frac{b^2}{a^2}\sqrt{\frac{2d}{g}}$; and the time that the surface takes to sink, from the depth d to any other depth d', is

$$\frac{b^2}{a^2}\left(\frac{\sqrt{d}-\sqrt{d'}}{\sqrt{\frac{1}{2}g}}\right).$$

Dan. Bernouilli, *Hydrodynamica.* sect. IV. § 13.; Bossut, *Hydrodyn.* tom. I. § 232.

The construction of the *clepsydra*, or water-clock, depends on this proposition. If the whole depth through which the surface of the water sinks in 12 hours be divided into 144 parts, it will sink through 23 of these in the first hour, 21 in the second, 19 in the third, and so on, according to the series of the odd numbers.

283. The vein of water, as it issues out, is contracted; and from that, and other causes, the actual discharge of water is not so great as it is computed to be in the foregoing theorems. This difference, however, affects only the absolute quantities, and not their proportions; the quantities really discharged being as the square-

roots of the depths multiplied into the areas of the orifices.

According to some very accurate experiments of Bossut, the actual discharge through a hole made in the side or bottom of the vessel, is to the theoretical as 1 to .62, or nearly as 8 to 5. The theoretical discharge, when computed, must, therefore, be diminished in this ratio to have the true one.

If the water issues, not through an aperture in the side, or bottom of the vessel, but through a pipe from 1 to 2 inches in length, inserted in the aperture, the contraction of the vein is prevented, and the actual discharge becomes to the theoretical, as 8 to 10, or as 4 to 5. In this way, therefore, the discharge is increased nearly in the ratio of 4 to 3. Bossut, § 523. See also Prony, *Architect. Hyd.* § 840.

The theoretical discharge, the discharge through an additional tube, and that through a simple perforation in the side, are as the numbers 16, 13, and 10 nearly.

284. Water thrown up in a perpendicular jet, ought to ascend to the height of the reservoir; but on account of the resistance of the air, the friction of the pipe, &c. it always falls short of that height; and it is found by experiment, that the differences between the heights of the jets and of the reservoirs, are as the squares of the heights of the jets themselves.

If H and H′ be the heights of two reservoirs, h and h' the heights of the actual jets,

$$H - h : H' - h' : : h^2 : h'^2.$$

This observation was made by MARIOTTE. BOSSUT, II. § 615.

The water ascends highest when the jet is not quite perpendicular; when it is perpendicular, the ascent is obstructed by the water falling back on the ascending column.

The height to which the water rises in the jet, is called the *height of the effective head.*

The parabola which an oblique jet describes, has its directrix not exactly as determined in § 281; but at the height of the effective head.

Of Conduit Pipes and Open Canals.

285. When the water from a reservoir is conveyed in long horizontal pipes, of the same aperture, the discharges made in equal times are nearly in the inverse ratio of the square roots of the lengths.

It is supposed that the lengths of the pipes to which this rule is applied are not very unequal. It is an approximation not deduced from principle, but derived immediately from experiment. BOSSUT, tom. II. § 647, 648. At § 673. he has given a table of the actual discharges of water-pipes, as far as the length of 2340 toises, or 14950 feet English.

286. Water running in open canals, or in rivers, is accelerated in consequence of its depth, and of the declivity on which it runs, till the resistance, increasing with the velocity, become equal to the acceleration, when the motion of the stream becomes uniform.

It is evident, that the amount of the resisting forces can hardly be determined by principles already known, and therefore nothing remains, but to ascertain by experiment, the velocity corresponding to different declivities, and different depths of water, and to try, by multiplying and extending these experiments, to find out the law which is common to them all.

The Chevalier du Buat has been successful in this research, and has given a formula for computing the velocity of running water, whether in close pipes, open canals, or rivers, which, though it may be called *empirical*, is extremely useful in practice. *Principes d'Hydraulique*, par M. Le Chevalier du Buat, 2 vol. 8vo, edit. 2. Paris 1786. Professor Robison has given an abridged account of this book, in his excellent articles on Rivers and Water-works, in the *Encyclopædia Britannica.*

Let V be the velocity of the stream, measured by the inches it moves over in a second; R a constant quantity, viz. the quotient obtained by dividing the area of the transverse section of the stream, expressed in square inches, by the boundary or perimeter of that section, *minus* the superficial breadth of the stream, expressed in linear inches.

The mean velocity is that with which, if all the particles were to move, the discharge would be the same with the actual discharge.

The line R is called by Du Buat the *radius*, and by Dr Robison the *hydraulic mean depth*. As its affinity to the radius of a circle seems greater than to the depth of a river, we shall call it, with the former, the *radius of the section*.

Lastly, Let S be the denominator of a fraction which expresses the slope, the numerator being unity, that is, let it be the quotient obtained by dividing the length of the stream, supposing it extended in a straight line, by the difference of level of its two extremities; or, which is nearly the same, let it be the co-tangent of the inclination or slope.

287. The above denominations being understood, and the section, as well as the velocity, being supposed uniform, $V =$

$$\frac{307\sqrt{R - \frac{1}{10}}}{S^{\frac{1}{2}} - \frac{1}{2}\log\left(S + \frac{16}{10}\right)} - \frac{3}{10}\sqrt{R - \frac{1}{10}};$$

or

$$\text{or } V = \sqrt{R - \frac{1}{10}}\left(\frac{307}{S^{\frac{1}{2}} - \frac{1}{2}\,\text{Log}\left(S + \frac{16}{10}\right)} - \frac{3}{10}\right).$$

See Du Buat, *ubi supra*, § 51., &c. At § 52. p. 65. the formula is reduced to English measure, as now given.

When R and S are very great, $V =$

$$R^{\frac{1}{2}}\left(\frac{307}{S^{\frac{1}{2}} - \frac{1}{2}\,\text{Log}\,S} - \frac{3}{10}\right) \text{ nearly.}$$

The logarithms understood here are the hyperbolic.

The slope remaining the same, the velocities are as

$$\sqrt{R - \frac{1}{10}}.$$

The velocities of two great rivers that have the same declivity, are as the square roots of the radii of their sections.

If R is so small, that $\sqrt{R - \frac{1}{10}} = 0$, or $R = \frac{1}{10}$, the velocity will be nothing; which is agreeable to experience; for in a cylindric tube $R = \frac{1}{2}$ the radius; the radius, therefore, equal two-tenths; so that the tube is nearly capillary, and the fluid will not flow through it.

The velocity may also become nothing, by the declivity becoming so small, that

$$\frac{307}{S^{\frac{1}{2}} - \frac{1}{2}\,\text{Log}\left(S + \frac{16}{10}\right)} - \frac{3}{10} = 0;\ \text{but}$$

if $\frac{1}{S}$ is less than $\frac{1}{500000}$, or than $\frac{1}{10}$th of an inch to an English mile, the water will have sensible motion.

288. In a river, the greatest velocity is at the surface, and in the middle of the stream, from which it diminishes toward the bottom and the sides, where it is least. It has been found by experiment, that if from the square root of the velocity in the middle of the stream, expressed in inches *per* second, unity be subtracted, the square of the remainder is the velocity at the bottom.

Principes d'Hydraulique, par Du Buat, § 67.

Hence, if the former velocity be $= v$, the velocity at the bottom $= v - 2\sqrt{v} + 1$.

289. The mean velocity, or that with which, were the whole stream to move, the discharge would be the same with the real discharge, is equal to half the sum of the greatest and

least velocities, as computed in the last proposition.

The mean velocity is, therefore, $= v - \sqrt{v} + \frac{1}{2}$.

This is also proved by the experiments of Du Buat.

290. Suppose that a river having a rectangular bed, is increased by the junction of another river equal to itself, the declivity remaining the same; required the increase of depth and velocity.

Let the breadth of the river $= b$, the depth, before the junction, d, and after it, x; and, in like manner, v and v' the mean velocities before and after; then $\frac{bd}{b + 2d}$ is the radius before, and $\frac{bx}{b + 2x}$ the radius after, so $v = \frac{307 R^{\frac{1}{2}}}{S^{\frac{1}{2}}}$, supposing the breadth of the river to be such, that we may reject the small quantity subtracted from R, in § 287.; and, in like manner, $v' = \frac{307 R'^{\frac{1}{2}}}{S^{\frac{1}{2}}}$. Then, substituting for R and R′, we have

$$v = \frac{307}{S^{\frac{1}{2}}} \times \sqrt{\frac{b\,d}{b+2\,d}},$$ and

$$v' = \frac{307}{S^{\frac{1}{2}}} \times \sqrt{\frac{b\,x}{b+2\,x}}.$$ Multiplying these into the areas of the sections $b\,d$ and $b\,x$, we have the discharges, viz. $b\,d\,v = \frac{307}{S^{\frac{1}{2}}} \times \frac{b\,d\sqrt{b\,d}}{\sqrt{b+2d}}$; and $b\,x\,v' = \frac{307}{S^{\frac{1}{2}}} \times \frac{b\,x\sqrt{b\,x}}{\sqrt{b+2x}}$. Now the last of these is double of the former; therefore $\frac{b\,x\sqrt{b\,x}}{\sqrt{b+2\,x}} = \frac{2b\,d\sqrt{b\,d}}{\sqrt{b+2\,d}}$, or $\frac{x^3}{b+2\,x} = \frac{4\,d^3}{b+2\,d}$, and $x^3 - \frac{3\,d^3}{b+2\,d}\,x = \frac{4\,b\,d^3}{b+2\,d}$, a cubic equation which can always be resolved by CARDAN's rule.

As an example, let $b = 10$ feet, and $d = 1$, then $x^3 - \frac{2}{3}\,x = \frac{10}{3}$, and $x = 1.4882$, which is the depth of the increased river. Hence we have $1.488 \times v' = 2\,v$, and $1.488 : 2 :: v : v'$, or $v : v' :: 37$ to 50 nearly.

When the water in a river receives a permanent increase, the depth and the velocity, as in the example above, are the first things that are augmented. The increase of the velocity increases the action on the sides and bottom, in consequence of which the width is augmented, and sometimes also, but more rarely, the depth. The velocity is thus diminished, till the tenacity of the soil, or the hardness of the rock, afford a sufficient resistance to the force of the water. The bed of the river then changes only by insensible degrees, and, in the ordinary language of Hydraulics, is said to be permanent, though in strictness this epithet is not applicable to the course of any river.

291. When the sections of a river vary, the quantity of water remaining the same, the mean velocities are inversely as the areas of the sections.

This must happen, in order to preserve the same quantity of discharge.

SECT. III.

PERCUSSION AND RESISTANCE OF FLUIDS.

292. If the sections of two streams be the same, the forces with which they strike on planes directly opposed to them, are as the squares of their velocities.

For the force of a stream must be as the force of each particle, and as the number of particles that strike in a given time. Now, the force of each particle is as the velocity of the fluid, and the number of particles that strike in a given time, the section being given, is also as that velocity. Therefore the whole force of the stream must be as the square of its velocity.

Hence if v be the velocity of any stream, a^2 the area of the section, f the force, and m a constant coefficient, to be determined by experiment, $f = m a^2 v^2$; or if h be the height belonging to the velocity v, so that $2 g h = v^2$, $f = 2 m a^2 g h$. The quantity $a g h$ denotes the weight of the column, of which the base is a^2, and the height h.

The constant quantity m is found to be different, according to the manner in which the fluid is received by the plane on which it strikes. When the plane is indefinite in extent, $m = 1$, so that $f = 2a^2gh$, or the force is double the weight of the column, of which the base is a^2, and the altitude h. This agrees with experiment, and has also been demonstrated by DAN. BERNOUILLI, *Comment. Petrop.* tom. VIII. p. 99., &c. See also *Encyclopædia Britannica*, 2d edit. Article *Resistance*, p. 109.

When the plane is no greater in extent than the section of the stream, $m = \frac{1}{2}$, and $f = a^2gh$, only half the former column.

Since, in two different streams, $f : f' :: a^2v^2 : a'^2v'^2$, if a^2v, or the quantity of water expended in a second, be the same in both, $f : f' :: v : v'$, or the forces are as the velocities simply.

293. If the plane struck by the stream be itself in motion, the impulse is as the square of the difference of their velocities.

If v be the velocity of the stream, and c the velocity with which the plane retires from it, then the force of percussion $= ma^2(v - c)^2$.

If c is opposed to v, then the force is $ma^2(v + c)^2$, and if $v = 0$, the force $= ma^2c^2$. The plane, therefore, moving against a fluid at rest, with the velocity c, suffers the same impulse as if the fluid were to move with the velocity 0, and the plane to remain at rest.

Hence the resistance of a fluid to a body in motion, is the same with the percussion of a fluid moving with the same velocity against the body at rest.

Though this conclusion is founded on reasoning very evident, yet, in fact, the resistance is less than the percussion, the velocity being in both cases the same, in the proportion of 5 to 6. Don Juan, in his *Examen Maritime*, makes this difference much greater. This arises, no doubt, from the action of the fluid on the hinder part of the body moving through it, by which the resistance is in some degree counteracted. It appears, however, to be only the absolute quantities, and not the ratios of the resistances, that are thus affected; the resistances being as the squares of the velocities.

In the reasoning on which this proposition is founded, we have supposed the only resistance to arise from the inertia of the particles. There is, however, another arising from the cohesion of these particles, which must be proportional to the quantity of that cohesion overcome in a given time, that is, to the velocity simply. This resistance is only perceived in slow motions, and is small in respect of the other, except in fluids of much viscidity. Coulomb has made experiments on this kind of resistance, *Mém. de l'Institut National*, tom. iii. p. 247, &c. He found the part of the resistance proportional to the velocity to be 17 times greater in oil than in water.

It must be remarked, that Don G. Juan makes the resistance which ships meet with proportional nearly, ly, *cæteris paribus*, to their velocities simply. *Examen Maritime*, tom. II. § 194, 195.

294. If a stream strike obliquely on a plane, its force is less than if it struck directly on the same plane, in the ratio of the square of the sine of the obliquity to the square of the radius.

If the area struck be a^2, v the velocity of the fluid, and i the inclination or the angle which the plane makes with the direction of the stream, the force $= \frac{v^2}{2g} . a^2 \sin i^2$.

This conclusion is not confirmed by experience, except when the angle i is so great, as to be between 60° and 90°. Bossut, *Hyd.* vol. I. § 375. It appears, from experiment, that the resistance to oblique planes consists of two parts, one of which is proportional to the square of the sine of i, and the other to a certain power (3.¼) of the angle i itself. Bossut found, that if a *prow*, in the form of a wedge, be drawn through a fluid, and if the complement of the angle which either face makes with the direction of the stream be called x, and if the resistance to the base of the wedge be 10000, the resistance to the wedge itself will be $10000 \times \cos x^2 + 3.153 \times \left(\frac{x^\circ}{6^\circ}\right)^{3\frac{1}{4}}$.

Bossut, *Hydrod.* tom. II. § 1019, & 1022.

This formula, however, is to be considered only as an approximation to the truth. It is entirely *empirical*,

cal, and does not even agree with the experiments from which it is deduced, when x is a very acute angle.

295. Supposing the resistance to oblique surfaces to vary, as the square of the sine of the inclination, the resistance to a sphere is half the resistance that would be made to the circumscribed cylinder moving in the direction of its axis.

NEWTONI *Prin. Math.* lib. II. prop. 34.

This proposition cannot be expected to hold accurately, as it is derived from a principle that is acknowledged to be erroneous. The propositions concerning the solid of least resistance, are faulty for the same reason. In general, though the theory of the perpendicular percussion and resistance of the dense fluids, agrees sufficiently with experiment for all practical purposes, that of their oblique percussion and resistance, as it stands at present, is not to be relied on. The experiments relative to percussion are more agreeable to theory than those that concern resistance. VINCE has found by experiment, that the force of the percussion of a fluid on an oblique plane, varies nearly as the sine of the inclination of the plane; but their resistance to a plane moving in a fluid, is not subject to any law that admits of a simple expression. *Encyc. Britan.* 3d edit. Article *Hydrodynamics*, p. 763.

SECT.

SECT. IV.

UNDULATION OF FLUIDS, OR THE FORMATION OF WAVES.

296. When the surface of water is unequally pressed on, in parts contiguous to one another, the columns most pressed on are shortened, and sink beneath the natural level of the surface, while those that are least pressed on are lengthened, and rise above that level.

As soon as the former columns have sunk to a certain depth, and the latter have risen to a certain height, their motions are reversed, and continue so, till the columns that were at first most depressed have become most elevated, and those that were most elevated, have become most depressed.

The alternate elevations and depressions thus produced, are called *waves.*

The water in the formation of waves has a vibratory or reciprocating motion, both in a horizontal and a vertical direction, by which it passes from the columns that are shortened to those that are lengthened,

ened, and returns again in the opposite direction. Progressive motion is not necessary to undulation.

297. The vibrations of water in the form of waves, may be compared to the reciprocations of the same fluid in a syphon or bent tube; and it was from this that NEWTON deduced the velocity of waves, and the time required to an undulation.

The time of an undulation, is the time from the wave being highest, at any point, to its being highest at that point again. The velocity of the wave is the rate at which the points of greatest elevation or depression seem to change their places.

298. If a be the altitude of a wave, b half the breadth, π the circumference of the circle, of which the diameter is 1, the time of an undulation is $\frac{\pi}{8}(a+b)^{\frac{1}{2}}$, and the space which the wave appears to pass over in a second, is $\frac{8\,b}{\pi\,(a+b)^{\frac{1}{2}}}$.

Examen Maritime, vol. I. § 816.

In these theorems, 8 is put for $\sqrt{2g}$, to which it is nearly equal.

If a be neglected, the velocity of the wave becomes $\frac{8\sqrt{b}}{\pi}$, which is the velocity as determined by Newton, *Principia*, lib. II. prop. 46. See also Bossut, *Hydrod.* tom. I. § 312.

299. While the depth of the water is sufficient to allow the oscillation to proceed undisturbed, the waves have no progressive motion, and are kept, each in its place, by the action of the waves that surround it. But if by a rock rising near to the surface, or by the shelving of the shore, the oscillation is prevented, or much retarded, the waves in the deep water are not balanced by those in the shallower, and therefore acquire a progressive motion in this last direction, and form breakers. Hence it is, that waves always break against the shore whatever be the direction of the wind.

Breakers formed over a great extent of shore, are distinguished by the name of *Surf*. The surf is greatest in those parts of the earth where the wind blows always nearly in the same direction.

Sect.

SECT. V.

HYDRAULIC ENGINES.

Engines moved by the IMPULSE *of Water.*

300. When a wheel is put in motion by a stream of water striking against a float-board, the action of the water diminishes, as the velocity of the wheel increases, till at last the momentum of the water, or of the accelerating force, is just equal to the momentum of the resistance, or of the retarding force. The motion of the wheel then becomes uniform.

Mills with undershot wheels, or such as receive the impulse of the water at right angles to their radii, are machines of this kind.

301. A machine driven by the impulse of a stream, produces the greatest effect, or does the most work in a given time, when the wheel moves with one-third of the velocity of the water.

This theorem was first given by PARENT, *Mém. de l'Acad. des Sciences*, 1704. See also MACLAURIN's *Fluxions*,

Fluxions, § 907. Euler, *Nov. Com. Pet.* tom. VIII. p. 230. Bossut, *Hydrod.* tom. I. § 390.

If c be the velocity of the stream, v that of the float-board, a^2 the section of the stream, or the area of the part struck, the impulse of the stream, by § 293. $= ma^2(c-v)^2$, and the effect or quantity of work done, in a given time, will be $ma^2(c-v)^2v$, which is a *maximum* when $v = \frac{c}{3}$.

The quantity $ma^2(c-v)^2$, or the moving force becomes, in this case, $ma^2 \times \frac{4}{9}c^2$, which, if m be determined as in § 292., is $= \frac{4}{9}a^2 \times \frac{c^2}{2g}, = \frac{8}{9}a^2h$, h being the height from which the water falls to acquire the velocity c. The load of the machine, that it may work to the greatest advantage, should be $\frac{4}{9}$ of that which is sufficient to prevent its motion.

The effect, when a maximum, is $\frac{8}{9}a^2h \times \frac{1}{3} = \frac{8a^2h}{27}$, that is, eight twenty-sevenths of the water expended.

If

In different falls of water, that act on wheels moving with a velocity in a given ratio to that of the water, if the expenditure be the same, the forces will be as the velocities simply, (§ 292.), and the effects of the machines driven by them as the squares of the velocities.

If the sections of two streams are the same, their forces will be as the squares of their velocities, or as the heights due to the velocity of the water, (§ 292.), and the effects of the machines driven by them, will be as the cubes of their velocities, or as the heights due to those velocities multiplied into their square roots.

These conclusions, all except the first, are remarkably verified by the experiments of Mr SMEATON. He found, that an undershot-wheel, when working to the greatest advantage, had a velocity which varied from one-third to one-half the velocity of the stream; and was, in great machines, nearer to the latter of these limits than the former. Also, that the moving power was nearly the whole column of water striking on the wheel; and we have shewn that it is eight-ninths of it.

When the undershot-wheel worked to the greatest advantage, its effect was about one-third of the water expended; we have found it to be $\frac{8}{27}$.

Mr SMEATON also found, that the expence of water being the same, the effect is as the effective head, or the square of the velocity. And, lastly, that when the section of the water is the same, the effect is as the cube of the velocity, both which we have deduced above. By a mistake, for which it is difficult to account, SMEATON supposed both these conclusions to be inconsistent with the theory of the percussion of fluids.

Machines moved by the Weight of Water.

302. When a wheel receives the water into buckets, at or near the highest point, it is put in motion by the weight of the water with which it is loaded on one side, and it is then called an *overshot* wheel.

303. An overshot wheel arrives at a state of uniform motion, when the momentum of the water in the buckets is equal to the momentum of the resistance.

The construction of the machine should be such, that when the motion of the wheel becomes uniform, the buckets may contain all the water of the stream, and may carry it down as low as possible before they allow it to escape altogether.

304. If the section of that part of the wheel to which the buckets are fixed be called c^2, and the perpendicular depth of the point where the water

leaves the wheel, below that where it enters, be called p, the moving force is a column of water equal to $c^2 p$; and as this force acts by a lever $= r$, its momentum $= c^2 p r$.

305. If A be the quantity of water issuing in a second, and h the height corresponding to the velocity of the circumference of the wheel, the effect of the machine, (supposing the wheel to receive the water, at or very near its highest point), is proportional to A $(2r - h)$.

ALB. EULER, *Enodatio Quæstionis quomodo vis Aquæ ad Molas circumagendas cum maximo lucro impendi possit*, 4to, Göttingen, 1754. BOSSUT, *Hydrod.* § 415, &c.

It is evident, that the effect will be the greater the less h is, or the less the velocity of the wheel; and as that velocity may be diminished indefinitely, $2r$ A is a limit to which the effect may approach nearer than any given quantity.

The power of the overshot-wheel is greater, *cæteris paribus*, than that of the undershot, nearly in the ratio of 27 to 8.

The maxim, however, that overshot wheels are the more powerful the slower they move, is found in practice to be subject to some limitations.

Mr SMEATON's experiments led him to conclude, that overshot wheels do most work when their circumferences move at the rate of 3 feet in a second, and that when they move considerably slower than this, they become unsteady and irregular in their motion. This determination is also to be understood with some latitude. He mentions a wheel 24 feet in diameter, that seemed to produce nearly its full effect though the circumference moved at the rate of 6 feet in a second; and another of the diameter of 33 feet, of which the circumference had only a velocity of 2 feet in a second, without any considerable loss of power. The first wheel turned round in 12″.6, the latter in 51″.9.

SMEATON, *Experimental Inquiry concerning the Natural Powers of Water and Wind to turn Mills*, &c. *Phil. Trans.* vol. LI. (1759,) p. 100, &c. See particularly p. 133, 134. Dr BREWSTER's *Notes on* FERGUSON's *Mechanics*.

Machines moved by the Re-action of Water.

306. Some machines are moved, not directly by the impulse of a stream of water, but indirectly, by the relief from pressure which the motion of the stream occasions.

The machine known by the name of BARKER's Mill is of this kind. Let ABGH, (fig. 23.), be a hollow cylinder, moveable about a vertical axis CD, and having the horizontal boxes EB, EG communicating with it, so that the whole may be filled with water from the top. The boxes have each an opening in the sides, opposite to one another, through which the water issues.

When these apertures are shut, and the whole filled with water, it remains *in equilibrio;* but if the apertures are opened, the columns having these apertures for their bases, and the depth of the water for their altitudes, will cease to press on the sides; and therefore the whole pressure on the sides where there are no perforations, will be greater than on the sides where the perforations are, by the sum of these two columns.

A moving force, therefore, equal to the pressure of these two columns, will impel the machine in a direction opposite to that in which the water issues, and in that direction the machine will begin to move.

307. The moving force in this instance becomes greater, after the machine has begun to move; for the water in the horizontal boxes acquires a centrifugal force, by which its pressure against the sides is increased. When the machine works to the greatest advantage, the centre of the perforations should move with the velocity $\frac{r}{r'}\sqrt{hg}$, where r is the radius of the horizontal arm, measured from the axis of motion to the centre of the perforation, and r' the radius of the perpendicular

tube, g being put for the force of gravity, or $32.\frac{1}{6}$ feet.

As $2\pi r$ is the circumference described by the centre of each perforation, $\frac{2\pi r'}{\sqrt{hg}}$ is the time of a revolution in seconds.

The quantity $\frac{r}{r'}\sqrt{hg}$ is also the velocity of the effluent water; therefore, when the machine is working to the greatest advantage, the velocity with which water issues is equal to that with which it is carried horizontally in an opposite direction; so that, on coming out, it falls perpendicularly down.

The effect of this machine is equal to A h, or the momentum of the water expended; so that it could, if there were no force lost by friction, raise up the whole of the water to the height from which it fell, and in an equal time. It is, therefore, in theory, the most advantageous application of water that is known; but, on account of the friction, which must be great, as the whole body of the water has to turn on the axis CD, (fig. 23.), it will probably be found in practice, that the overshot wheel is, in many cases, preferable. In the construction of this machine for practical purposes, the water is made to flow in pipes along a conical surface, diverging downwards, and is discharged through several apertures.

On the construction of this machine, see EULER, *Mém. de l'Acad. de Berlin*, tom. VI. 1750, p. 311.; and, again, tom. VII. p. 271. ALB. EULER, *Enodatio*, &c. § 64. BOSSUT, *Hydrod.* tom. I. § 424. BREWSTER'S *Notes on* FERGUSON'S *Lectures*, vol. II. p. 205.

AEROSTATICS.

AEROSTATICS.

SECT. I.

OF HEAT.

In treating of Aëriform Fluids, we begin with the consideration of Heat, which, though it affect all material substances, acts more remarkably on elastic fluids than on other bodies. We shall treat of heat as affecting the Bulk, the Fluidity, and the Elasticity of bodies.

308. HEAT OR CALORIC may be regarded as a substance that penetrates and expands all bodies, and that produces in us the sensations of heat and cold.

Though we conceive heat to be a *substance*, it is never found in an independent state, or existing otherwise than as a property of body.

309. As heat is known to us principally by its power of expanding bodies, we may take the expansion of some known substance as a measure of the variations of heat.

The substance selected for measuring heat is Mercury; and the Thermometer is an instrument indicating the variations of bulk to which we conceive the variations of heat to be proportional.

It has been found, that the variation of the bulk of mercury, from being immersed in freezing, and in boiling water, bears always the same proportion to the whole bulk; and, therefore, that variation, divided into a certain number of parts, may be taken as a scale of the degrees of heat.

In Fahrenheit's thermometer, the interval above named is divided into 180°, and numbered at the lower end 32°, and at the upper end 212°. In the Centigrade Thermometer, the above interval is divided into 100°.

The word *temperature* is used to denote the heat indicated by any degree of the thermometer.

310. All bodies, whether solid or fluid, are expanded by heat; but not all in the same proportion.

Expansion of different Bodies for 180° *of* FAHRENHEIT, *viz. from* 32° *to* 212°.

	IN BULK.	IN LENGTH.
Gold, -	.0042 $= \frac{1}{238}$	.0014 $= \frac{1}{714}$
Platina, -	.0026 $= \frac{1}{385}$	.00085 $= \frac{1}{1155}$
Silver, -	.006 $= \frac{1}{160}$	.003 $= \frac{1}{480}$
Copper, -	.0051 $= \frac{1}{196}$	.0017 $= \frac{1}{588}$
Brass, -	.0056 $= \frac{1}{178}$	.00153 $= \frac{1}{533}$
Zinc, -	.0093 $= \frac{1}{108}$	.0031 $= \frac{1}{322}$
Mercury, -	.02 $= \frac{1}{50}$	.0066 $= \frac{1}{150}$
Water, -	.0466 $= \frac{1}{21.5}$	.0155 $= \frac{1}{64}$
Salt waters,	.05 $= \frac{1}{20}$	.0166 $= \frac{1}{60}$
Alcohol, -	.01 $= \frac{1}{100}$	.0033 $= \frac{1}{300}$
All gases, or permanently elastic fluids,	.376 $= \frac{1}{2.66}$	.1253 $= \frac{1}{7.98}$

See DALTON's *New System of Chemical Philosophy,* vol. i, p. 44.

311. At the heat of —40° mercury freezes, and at 600° it boils, and is in both cases useless as a measure of heat. For measuring great cold, a thermometer of alcohol or spirit of wine is employed; for measuring great heat, the expansion of a solid is used, and the instrument is called a *Pyrometer.*

The best pyrometer hitherto in use, is that invented by Mr Wedgwood, consisting of a small cylinder of Cornish clay, which contracts continually by heat. The scale of this instrument begins at a red heat, of such intensity as to be visible in day-light, which, on Fahrenheit's scale, could it be continued so far, would be marked by 1077°. From this it ascends, (by degrees that are each of them equal to about 130° of Fahrenheit,) as far as 240°, or 32277° of Fahrenheit, which is its utmost limit; 28° marking the melting heat of silver, 32° of gold, 125° the greatest heat of a glass-house furnace, and 150° the temperature at which pig-iron melts.

312. The fact that all bodies expand by heat, admits also of another exception. Water is most dense at a temperature between 37° and 39°, and its specific gravity diminishes both by the addition and subtraction of heat.

When water is converted into ice a new arrangement takes place, and the specific gravity is considerably diminished.

313. Heat diffuses itself on all sides, and passes continually from bodies in which the temperature is greater, to those in which it is less; and if a body be placed in a medium of a temperature different from its own, the momentary variations of its temperature will be as the differences between the temperature of the body and of the medium; so that if the times, reckoned from any instant, be taken in arithmetical progression, the variations of temperature in the body, and also the differences between its heat and the heat of the medium, will decrease in geometrical progression.

This law of the heating and cooling of bodies, was first taken notice of by Newton, *Scala Graduum Caloris, Phil. Trans.* (1701,) N° 270. It was afterwards proved by the experiments of Professor Richman of St Petersburg. *Novi Comment. Petrop.* tom. i. (1747, 1748,) p. 174., &c.

It follows from it, that if D be the difference between the temperature of the medium, and of the body placed in it, and if $\frac{D}{n}$ be the heat lost in one second; then, at the end of 1″, 2″, 3″, &c. the heats are $D\left(1-\frac{1}{n}\right)$, $D\left(1-\frac{1}{n}\right)^2$, $D\left(1-\frac{1}{n}\right)^3$, &c.; and the decrements of heat in each second $\frac{D}{n}$, $\frac{D}{n}\left(1-\frac{1}{n}\right)$, $\frac{D}{n}\left(1-\frac{1}{n}\right)^2$, &c.; and, therefore, if t

be the time in seconds, reckoned from the moment when the difference of the temperatures was D; and if Δ be the remaining difference of temperature at the end of the time t, we have $D\left(1-\frac{1}{n}\right)^t=\Delta$,

and $$t=\frac{\log. D-\log. \Delta}{\log. n-\log. (n-1)}.$$

If this law is rigorously observed, no body can ever perfectly acquire the temperature of the medium in which it is placed, and this may be the fact, speaking with mathematical exactness, though the difference of the two temperatures may soon become so small as to escape observation. In respect of us, the bodies are then of the same temperature.

314. The diffusion of heat through a fluid, is promoted by a hydrostatical principle, the heat rarifying the fluid, and so producing a motion and mixture of parts, by which the heat is communicated more rapidly through the whole, than it could be through a mass of which the parts were immovable in respect of one another.

The communication of heat in this way, is so rapid, that it renders the ordinary slow process of acquiring and losing heat by contact almost imperceptible, and has given rise to the error, that heat has no tendency to pass through fluids, except in consequence of the mixture of their parts.

315. Heat escapes from bodies that are heated above a certain temperature, by *radiation*, or by passing in straight lines through the air with great rapidity.

The heat thus emitted from bodies, is reflected from the surfaces of metallic specula, like the rays of light.

A body heated, though not so as to shine, and placed before a concave speculum of metal, communicates heat instantaneously to a thermometer in the corresponding focus. A cold body does the same; and it is remarkable, that an effect so difficult to be explained, is, nevertheless, perfectly consistent with the law of continuity. The experiment of the reflection of cold was first made by M. PICTET of Geneva. *Essai sur le Feu, par* M. A. PICTET, § 69.

Radiant heat from a luminous body, is refracted as well as reflected; the rays from a candle or a fire produce heat in the focus of a lens. It does not appear that the same happens in the case of obscure heat, except when it comes from the sun, which, according to the experiments of Dr HERSCHEL, is concentrated in the focus of a lens. *Phil. Trans.* 1803, p. 298, 304.

Dr HERSCHEL, in a series of very interesting experiments, separated the caloric from the light of the sun's rays, and shewed that the focus of the former was not the same with that of the latter. *Phil. Trans. ibid.*

316. The heat propagated by radiation from different bodies, varies with the nature of their external surfaces, the quantity that flows in a given time from a body with a polished surface, being much less than would flow from the same body with a rough surface.

This was discovered by Professor **Leslie**, and proved by a variety of curious experiments. *Experimental Inquiry into the Nature and Propagation of Heat*, p. 17, &c.

When the fluid contained in a vessel, is intended to retain its heat long, the vessel should be of metal, and its surface smooth and bright.

If the fluid is required to cool as fast as possible, the surface should be rough, covered with paper, charcoal, &c. **Dalton's** *New System of Chemical Philosophy*, Part I. p. 116.

317. If two portions of the same fluid, of which the masses are M and M′, and the temperatures t and t', be mixed together, the temperature of the mixture will be $\frac{Mt + M't'}{M + M'}$.

This formula is investigated, on the supposition that no part of the heat is lost by the mixture.

On making the experiment, the results are not found to agree exactly with those deduced from the formula. The expansions of mercury, therefore, do not appear to observe the same law with the variations of temperature; and it is found, that, for equal variations of temperature, the variations of the bulk of the mercury constitute a series, of which the differences are in arithmetical progression, or in which the second differences are constant. Hence the relation between the temperature and the expansion of mercury, may be expressed by a quadratic equation, or by the abscissæ and ordinates of a line of the second order.

318. If D be any number of degrees reckoned from the freezing point of water in FAHRENHEIT's thermometer, the real temperature corresponding to D is $207\left(-1+\sqrt{\frac{D}{72}+1}\right)$.

This formula is derived from DALTON's experiments. See his *System of Chemical Philosophy*, Part I. p. 14.

If D = — 72, which is known to mark the degree according to FAHRENHEIT's thermometer, at which mercury congeals; the temperature, as deduced from this rule, is — 207, which, therefore, is the real distance between the freezing of mercury and the freezing of water. Hence the degrees on the mercurial thermometer, reckoned from the degree which marks the congelation of mercury, are as the squares of the real temperatures, reckoned from the same point of congelation.

The correction of the mercurial thermometer, arising from this law, has not yet been employed; but the reality of it seems to be sufficiently established.

319. When fluids of different kinds, and of different temperatures, are mixed, as above, the temperature of the mixture is very far from corresponding with the theorem in article 317, in so much as to make it certain, that equal bodies, for equal differences of temperature, do not contain equal quantities of heat. Beside the masses M and M′, therefore, we must introduce two indeterminate co-efficients c and c', before the formulas can be applied; we have then

$$t = \frac{cM + c'M'}{c''(M + M')}$$

for the temperature of the mixture.

In this formula, c and c' are the numbers that denote the differences of heat contained in two equal portions of these fluids, for equal differences of temperature. They may be said to denote the capacities of the bodies for heat; and their values must be determined by experiment.

320. When equal weights of water and ice are put into the same vessel, the water being at the temperature 176°, and the ice at 32°, the ice is in-

stantly melted, and the temperature of the whole is found to be exactly 32°.

In this experiment, 140 degrees of heat have entirely disappeared; and the same happens whatever be the temperature of the water. Dr BLACK was the first who made this experiment; and as he found that the heat which had thus ceased to affect the thermometer, was not lost, but became sensible again on the congelation of the water, he said that it had become *latent*,—a term well adapted to express the fact, without any allusion to theory.

321. From the combination of this experiment with the formula in the last article, the beginning of the scale of heat, or the point to which the mercury in the thermometer would sink, if it were to lose all its heat, may be determined; and it will be found, that if c is the capacity of water for heat, c' of ice, and x the number of degrees that the beginning of the scale of heat is below the beginning of FAHRENHEIT'S, $x = \frac{32\,c' + 112\,c}{c - c'}$.

For, according to this notation, the total heat in ice, of the temperature 32°, is $c'(32 + x)$; and the heat in water of the temperature 176°, is $c(176 + x)$; therefore the total heat, when they are mixed, and the whole reduced to water, is $2\,c\,(32 + x)$. This last expression must be equal

to

to the sum of the two former, or

$$32\,c' + 112\,c = c\,x - c'\,x, \text{ and}$$

$$x = \frac{32\,c' + 112\,c}{c - c'}.$$

If c be to c' as 10 to 9, which appears, from experiment, to be true, $x = 288 + 1120 = 1408$; this, therefore, is the distance of the *zero* of temperature from the *zero* of FAHRENHEIT, and its distance below the freezing of water, is, therefore, 1440°.

In this investigation, it is supposed, that, however low the ice is cooled, c' remains of the same value.

322. When water is put in a vessel on the fire, it continually receives heat, and its temperature increases, till, at a certain point, the water boils, and rises in the form of vapour, after which the temperature of the water increases no farther, as much heat being carried off every instant by the vapour, as is received from the fire.

The temperature which water then attains, as was before remarked, is nearly constant, and is the same in all cases, except in as far as it is affected by the pressure of the atmosphere, in the manner about to be explained. The vapour thus formed, is an invisible elastic fluid, of a temperature not less than 212 degrees. A great quantity of heat passes into the latent state, when vapour or steam is generated. This was also discovered by Dr BLACK.

323. On the different quantities of heat, therefore, united to the substance that we call water, depends its existence in the state of a solid, a liquid, or an elastic fluid. In passing through these states in the order just mentioned, a great increase of temperature is not only necessary, but a great quantity of heat must be absorbed, or, from being sensible, must become latent.

This is true not only of water, but of mercury, oil, and probably of all unelastic fluids. It perhaps holds of all bodies that are melted by heat, and it may be true even of those fluids that, like air, appear to be permanently elastic.

324. Since heat is absorbed when a solid passes into a liquid state, whatever produces liquefaction produces a diminution of sensible heat; and this, in the case of what are called Freezing Mixtures, can be employed to produce a cold of —90°. So, also, cold is produced by evaporation, or the conversion of an incompressible into an elastic fluid.

325. Water boils with less heat, when the weight of the air diminishes, and, conversely, requires a greater heat to make it boil, when the pressure of the air is increased.

On going to the top of a high mountain, where the weight of the air is diminished by one-tenth, the boiling point of water is somewhat less than 207°. So, on going into a deep mine, where the weight of the air is increased by one-thirtieth, the boiling point of water is nearly 214°.

If, by means of the air-pump, the pressure of the air be reduced to one-half, water will boil nearly at the temperature of 180°.

326. The elasticity of the steam generated from water, is measured by the weight that compresses it; that is, by the weight of the atmosphere pressing on the surface of water; and the relation between this force and the heat required to produce the steam, is expressed by the following exponential equation, where x is the heat measured from the ordinary boiling point, or 212°, and f is the force of compression measured in inches of mercury,

$$f = 30\left(1.235 - .015 \times \frac{4x}{45}\right)^{\frac{4x}{45}}.$$

This formula is deduced from Dalton's *Experiments*, *Manchester Memoirs*, vol. v. p. 555., &c. and nearly coincides with the method which he used for interpolating between the numbers determined by observation: x is positive when the heat is greater than 212°, negative when less.

For the purpose of calculation, it is best to take the logarithm of the above expression; it is then

$$\log f = \log 30 + \frac{4x}{45} \log \left(1.250 - .015 \times \frac{4x}{45}\right).$$

Hence, when x is known, f may be found, which measures both the compression on the surface of the water, and the elastic force of the steam. The mean weight of the atmosphere is here understood to be equal to 30 inches of mercury, the mean height of the barometer at the level of the sea. In the construction of thermometers, care should be taken to mark the boiling point, or 212, only when the barometer stands at 30 inches.

The above formula will serve to calculate the amount of the corrections to be made, when the barometer stands at any other height. The same may be determined from Mr DALTON's Table, *ubi supra*, p. 561., or simply, as the variations are but small, by allowing at the rate of 1° for six-tenths of an inch in the height of the barometer.

327. It appears, from the experiments of Mr DALTON, that the above formula applies to the steam produced from all liquids; providing x be reckoned from the ordinary boiling point of the liquid, or that at which it boils when the barometer stands at 30 inches.

Manchester Memoirs, vol. v. p. 564. MURRAY's *Chemistry*, vol. I. p. 234.

328. In permanently elastic fluids, the law that connects the temperature, and the elasticity, is simpler than in the preceding instances; for when the temperature increases in arithmetical, the elasticity increases in geometrical progression, and the difference between the temperature of freezing water and of boiling, increases the elasticity and the bulk of such fluids from 1 to 1.376.

If, therefore, the elasticity of such a fluid, at any temperature, be called 1, and if the number of degrees above that temperature be x, and f the elastic force, $f = (1.376)^{\frac{x}{180}}$.

In logarithms, $\log f = \frac{x}{180} \times \log (1.376)$.

Dalton's *New System of Chemical Philosophy*, part i. p. 44.

When the variations of heat are small, or not exceeding the ordinary vicissitudes of heat and cold in the atmosphere, the variations of elasticity may be supposed proportional to them. It is usual to suppose, that for a degree of Fahrenheit, the elasticity is increased or diminished by .00243 of the whole, or by $\frac{1}{411.5}$.

In this estimation of the elastic force of fluids, the thing actually measured is the expansion produced by the heat, while the compressing force remains the same. The force that would be required to reduce the fluid into its former dimensions, the temperature being the same, is the direct measure of the increase of elasticity. On the supposition, however, that when the temperature is given, the elasticity is inversely as the space which the fluid occupies, the absolute increase of the elasticity, produced by an increase of temperature, is proportional to the expansion. This supposition will be shewn to hold of air and other elastic fluids.

329. The law by which the elasticity and the expansion of aeriform fluids are regulated, is different from the law by which the elasticity and expansion of vapour are regulated, while the latter remains in contact with the liquid from which it is generated; but when the communication between them is cut off, the expansion by heat, and the increase of elasticity in the two kinds of fluids, follow the same law. The expansion is at the rate of 1.376 to 1, for 180 degrees of FAHRENHEIT; and for any other temperature x, is

$(1.376)^{\frac{x}{180}}$, or $= (1.376)^{\frac{1}{180}} \times (1.376)^x$, that is, $= (1.0018)\ (1.376)^x$.

This has been found by the experiments of GAY LUSAC.

330. Heat is mechanically produced by Percussion and Friction.

This holds both of solids and of fluids. A nail can be made red hot by hammering, and fire may be produced by the friction of one piece of timber against another. COUNT RUMFORD made water boil by the boring of a cannon.

The condensation of air produces heat: tinder may be lighted, and gunpowder set fire to, if air be suddenly compressed by the stroke of a piston.

331. The greater part of the preceding facts may be explained on the hypothesis, that heat or caloric is an elastic fluid *sui generis*, the particles of which repel one another, but are strongly attracted, though in different degrees, by the particles of all other bodies.

1. The introduction of this fluid into bodies, must increase their volume, and an equilibrium in this fluid will take place over a certain extent, when the attraction of the bodies within that space, for the caloric contained in it, is able to balance the repulsion of the particles of the caloric for one another, and this will happen when the caloric is distributed among the bodies, in proportion to the forces with which they attract it. Hence there may be an equilibrium in the caloric of bodies, or a sameness of temperature, though the quantities of caloric be very different; and from this proceeds the different capacities of bodies for heat, and the conversion of sensible into latent heat, or the contrary, when those capacities are changed.

2. Caloric converts solids into liquids, by overcoming their cohesion so far, that the particles attract one another with forces that are as a certain function of the distances, (and of the distances only,) of the particles from one another. This produces the equilibrium of the particles, or their tendency to obey the action of any force in any direction. The addition of more caloric, so as entirely to overcome the attraction of the particles for one another, converts the liquid into an elastic or aeriform fluid, the particles of which, by means of the heat with which they are combined, are made to repel one another.

3. Heat must escape from bodies, both in consequence of the attraction of those bodies that have less heat, and the mutual repulsion of the parts of caloric. When the temperature of a body is raised to a certain height, this last cause may operate without the former. On this account, the caloric in the air is impelled from the radiant body on all sides, and is condensed on the surfaces of the adjacent bodies, which, by their attraction, resist its motion. It is probable that this is effected by undulations, propagated through the calorific fluid. The apparent reflection of cold may also receive a solution on this hypothesis.

4. The production of heat by percussion and friction, is more difficult to be explained on this hypothesis than any of the other phenomena.. If we suppose the temperature, or the expansion of a body, to be affected, not merely by the quantity of caloric con-

tained in it, but by the state of movement or vibration in which the caloric exists, (which state percussion and friction may very much determine,) this appearance will also admit of a solution.

SECT. II.

EQUILIBRIUM OF ELASTIC FLUIDS.

332. THE fundamental theorems demonstrated in the first section of Hydrostatics, are common to the compressible and the incompressible fluids.

See articles 243, 244, &c.

333. As Atmospheric Air is the best known of all elastic fluids, we shall take it for the subject of our investigation, defining it, A heavy and elastic fluid, which resists compression with forces that are directly as its density, or inversely as the spaces within which the same quantity of it is contained.

The accuracy of this definition is known from experience.

The weight of air is known from the Torricellian experiment, or that of the barometer. The air presses on the orifice of the inverted tube with a force just equal to the weight of the column of mercury sustained in it.

A bottle, weighed when filled with air, is found heavier than after the air is extracted. The pressure of the atmosphere is about 14 lbs. on every square inch of the earth's surface. Hence the total pressure on the convex surface of the earth = 10,686,000,000 hundreds of millions of pounds.

The elastic force of the air is proved, by simply inverting a vessel full of air in water: the resistance it offers to farther immersion, and the height to which the water ascends within it, in proportion as it is farther immersed, are proofs of the elasticity of the air contained in it.

When air is confined in a bent tube, and loaded with different weights of mercury, the spaces it is compressed into are found to be inversely as those weights. But those weights are the measures of the elasticity; therefore the elasticities are inversely as the spaces which the air occupies.

Now the densities are also inversely as those spaces; therefore the elasticity of air is directly as its density. This law was first proved by MARIOTTE's experiments.

In all this, the temperature is supposed to remain unchanged.—These properties seem to be common to all elastic fluids.

Air resists compression equally in all directions. No limit can be assigned to the space which a given quantity of air would occupy if all compression were removed.

334. As we ascend from the surface of the earth, the density of the air necessarily diminishes: For, each stratum of air is compressed only by the weight of those above it; the upper strata are therefore less compressed, and of course less dense, than those below them.

335. Supposing the same temperature to be diffused through the atmosphere, required the law by which the density and elasticity of the air diminish as the height above the surface increases.

Let D, D', D'', be the densities of three contiguous strata of air, each of a given thickness, so small, that the density of the air in the whole space may be regarded as constant. Let W be the weight of the incumbent air pressing on the lowest of these strata; W' that pressing on the next, and W'' that pressing on the third, and let m be a co-efficient, such that $D = mW$; then, because the densities are as the compressions, $D' = mW'$, and $D'' = mW'$.

Therefore $D - D' = m(W - W')$, and

$$D' - D'' = m(W' - W'').$$

But $W - W'$ is just equal to the weight of the first of the three strata, and $W' - W''$ to that of the second; therefore $D - D' : D' - D'' :: D : D'$, and multiplying extremes and means $DD'' = D'^2$. The density of the middle stratum, is therefore a mean proportional between the densities of the other two, and whatever be the number of equidistant strata, their densities are in continual proportion.

336. If, therefore, the heights from the surface be taken increasing in arithmetical progression, the densities of the strata of air will decrease in geometrical progression. Also since the densities are as the compressing forces, that is, as the columns of mercury in the barometer, the heights from the surface being taken in arithmetical progression, the columns of mercury in the barometer at those heights will decrease in geometrical progression.

As logarithms have, relatively to their numbers, the same property, therefore if b be the column of mercury in the barometer at the surface, and β at any height h above the surface, taking m for a constant co-efficient, to be determined by experiment,

$$h = m(\log b - \log \beta), \text{ or } h = m \log \frac{b}{\beta}.$$

It is evident, that m may be determined by finding trigonometrically the value of h in any case, where b and β have been already ascertained. Now, when this is attempted in different states of the atmosphere, it is found that the values of m are the same when the mean temperature of the columns of air intercepted between the stations at which the observations are made, is the same; but that the values of m are different, when the mean temperatures are different. It has been found, that when h is to be expressed in English fathoms, and the common tabular logarithms are employed, and if the mean temperature be that of freezing, $m = 10000$, so that the difference of the tabular logarithms, counting the first four places integers, gives the height in fathoms.

For other temperatures, the value of m involves the expansion of air for one degree of heat, which has been taken at .00243, or .00244, as determined by the experiments of General Roy and Sir George Shuckburgh. A correction is also required for the temperature of the mercury itself; for that being rarely the same in the two barometers, a reduction is necessary to bring them both to one temperature.

337. If, therfore, b be the height of the mercury in the barometer at the lowest station, β at the highest, t and t' the temperatures of the air at those stations, f the fixed temperature at which no correction is required for the temperature of the air; and if q and q' be the temperatures of the quicksilver in the two barometers, and n the ex-

pansion of a column of quicksilver, of which the length is 1, for 1° of heat; h being the perpendicular height (in fathoms) of the one station above the other,

$$h = 10000\,(1 + .00244\left(\frac{t+t'}{2} - f\right) \log \frac{b}{\beta \times (1 + n\,(q - q')},$$

n being nearly $= \frac{1}{10000}$.

If the centigrade thermometer is used, because the beginning of the scale agrees with the temperature f, so that $f = 0$, the formula is more simple; and if the expansion for air and mercury be both adapted to the degrees of this scale,

$$h = 10000\,(1 + .00441\left(\frac{t+t'}{2}\right) \log \frac{b}{\beta(1 + .00018(q - q')}.$$

338. La Place has given a formula for barometric measurement somewhat different from the preceding. If t and t' be the temperatures according to the centigrade thermometer, all other things remaining as above, the height in metres, or

$$h = 18393\,\left(1 + \frac{2\,(t+t')}{1000}\right) \log \frac{b}{\beta + \frac{q - q'}{.5412}\,\beta}.$$

In this formula, the expansion for the temperature of one degree is .004; according to the formula of Sir George Shuckburgh and General Roy, it is .00441.

The former expansion agrees very well with the rate and law of expansion, stated in the last section, from the experiments of Dalton and Gay Lusac. The expansion in the English formula is therefore probably too great, though the results deduced by means of it agree tolerably with observation.

The true expansion, however, may be introduced, by increasing the constant multiplier by 49, or which is nearly the same, by $\frac{100}{2}$.

The formula, to give h in English measure, and to employ the true expansion, using the centigrade thermometer, will be

$$h = 10050\left(1 + \frac{2(t + t')}{1000}\right) \log \frac{b}{\beta(1 + .00018(q - q'))}.$$

The addition to be made on account of the 50 added to the constant multiplier, is very easy, as it only requires the difference of the logarithms, or the approximate height, to be increased by a two hundredth, or the half of a hundredth part.

The fact that the mercury in the barometer sinks on ascending into the atmosphere, was first suggested by Pascal, and ascertained by experiments under his direction. The application of this to the measurement of heights, was made in France by Mariotte, and in England by Halley. It remained, however, very imperfect, till De Luc applied the corrections for heat, and directed simultaneous observations to be made at the top and bottom of the mountain. He fell into some errors, which were pointed out by Trembley, and afterwards corrected by the observations and experiments of Sir George Shuckburgh and General Roy. La Place has, more lately, made barometrical measurement the subject of his researches, and has given the formula mentioned above.

See De Luc, *Recherches sur les Modifications de l'Atmosphere.* Horsely and Maskelyne, *Phil. Trans*, vol. lxiv. p. 214, & 158. Trembley, *apud* Saussure, vol. ii. p. 616. Damen, *Dissertatio Phys. & Math. de Montium Altitudine, Barometro metienda*, Hague, 1783. General Roy, *Phil. Trans.* vol. lxvii. (1777,) p. 653, &c. Sir George Shuckburgh, *ibid.* 513, &c. La Place, *Mechanique Celeste*, tom. iii. p. 289.

Though much has been done by the authors just named, the barometrical method of measuring heights is probably not yet so perfect as it may be rendered. Its exactness depends on the comparison of trigonometrical measures with barometrical observations. Now, the trigonometrical measures of heights are subject to considerable inaccuracy, from the variations of refraction, unless the refraction itself be observed at the same time with the angles of elevation. If this is done, or if the height be taken by levelling, the result may be obtained with great precision. The number of instances, however, in which either of these precautions has been used, is not great enough to afford an exact determination of the constant quantities that enter into the barometric formula. It would be useful also to ascertain by more accurate experiments, the law according to which the temperature of the air diminishes on ascending from the surface of the earth.

339. The temperature of the air diminishes on ascending into the atmosphere, both on account of the greater distance from the earth, the principal source of its heat, and the greater power of absorbing heat that air acquires, by being less compressed.

The heat of the atmosphere appears to decrease in a less ratio than the distance increases. M. DE SAUSSURE found, that by ascending from Geneva to Chamouni, a height of 347 toises, REAUMUR's thermometer fell 4°.2, and that on ascending from thence to the top of Mont Blanc, 1941 toises, it fell 20°.7; this gives 221 feet English for a diminution of 1° of FAHRENHEIT in the first case, and 268 in the second. Nevertheless, from the accuracy which the rule for barometrical measurement possesses, it may be inferred, that the decrease of heat for the greatest heights which we can reach, is not far from uniform; but that the rate for any particular case must be determined by observation, though the average in our climate may be stated at 1° for 270 feet of perpendicular ascent. On this subject, see La Grange, *Mém. de Berlin*, 1772, p. 206., &c. He thinks that the hypothesis of a uniform decrease of heat is the most conformable to appearances.

Euler, in a volume of the same Memoirs, for 1754, p. 140, considers a harmonical progression as the most probable. If the sole cause of the diminution of temperature were distance from the earth, and if it were admitted that there is no current of air perpendicularly upwards, as there certainly is not, the diminution of temperature would follow the inverse ratio of the distance from the centre of the earth. *Transactions of the Royal Society of Edinburgh*, vol. vi. p. 365.

Professor Leslie, in the Notes on his *Elements of Geometry*, p. 495. edit. 2d, has given a formula for determining the temperature of any stratum of air when the height of the mercury in the barometer is given. The column of mercury at the lower of two stations being b, and at the upper β, the diminution of heat, in degrees of the centigrade, is $\left(\frac{b}{\beta}-\frac{\beta}{b}\right)25$. This seems to agree well with observation.

340. If the atmosphere were reduced to a body of the same density which it has at the surface of

the earth, and of the same temperature, the height to which it would extend, is in fathoms, equal to $4343\left(1 + .00441\,\frac{t + t'}{2}\right)$, or, taking the expansion as in art. 338. $= 4343\left(1 + \frac{4 \times t}{1000}\right)$:

Hence if b be the height of the mercury in the barometer, reduced to the temperature t, the specific gravity of mercury is to that of air, as b to $4343\left(1 + \frac{4t}{1000}\right)$, or the specific gravity of air $= \dfrac{b}{72\,(4343)\left(1 + \frac{4t}{1000}\right)}$. The divisor 72, is introduced in consequence of b being expressed in inches.

341. Let a balloon, of a given cubical content, in feet, c^3, be filled with hydrogen gas, 13 times as elastic, and therefore 13 times rarer than common atmospheric air, under the same compression; and let it also be loaded with a given weight W; required to what height it will ascend.

If y be the specific gravity of any stratum of air, in which the balloon is placed, that of water being unity, since a cubic foot of water weighs 62.48 lbs. avoirdupois, $c^3 \times (62.48)\,y$ is the weight of the air displaced, or the whole force tending to support the balloon. But the whole weight of the balloon is $W + (62.48)\, c^3 \times \frac{y}{13}$, and therefore when the balloon is just supported, and has no tendency to ascend, $62.48 \times c^3 y = W + \frac{62.48}{13} c^3 y$, or

$$y = \frac{13\,W}{874.72\,c^3}.$$

When the density of the stratum where the balloon will float is found, the density at the surface being computed by the last article, the height will be determined by the rules for barometric measurement.

PNEUMATICS.

342. THE doctrine of Pneumatics, treats of elastic fluids, as accelerating or retarding the motion of bodies; of air, as the vehicle of sound; and, lastly, of air, as the vehicle by which heat and moisture are distributed over the surface of the earth. It is, therefore, naturally divided into three sections.

SECT. I.

AIR AS ACCELERATING OR RETARDING MOTION.

Machines for raising Water.

343. IF a bent tube, open at both ends, have the end of the shorter leg immersed in water; and

if the air in the tube be sucked out, or otherwise extracted, the water will flow through the tube in a continued stream, providing the height from the surface of the water, to the top of the bending, does not exceed 32 or 33 feet.

A bent tube of this kind is called a *syphon*. If the syphon, instead of having the air extracted by suction, be filled with water, and the ends stopped till it is inverted, with the shorter leg immersed in water, the same effect will follow.

It is the greater weight of the water in the longer leg of the tube that determines the fluid to descend on that side, and the pressure of the air continues the supply.

The syphon may be employed to raise water over a height less than 33 feet, if it is to descend below the level of the fountain on the opposite side. It could not, however, be conveniently applied, if the height was near to 33 feet; because the velocity with which the water in the shorter leg is made to ascend, depending on the difference between that height and 33 feet, the length of the column which the air is just able to sustain, might not afford a sufficient supply for the stream descending in the longer leg.

The Syphon is properly a pneumatico-hydraulic machine, the action of water and of air being both necessary to its effect. It is, however, the simplest of all the instruments employed in raising water.

344. The Air-pump is a machine by which the air is extracted from close vessels. A cylindric barrel in which a piston works, communicates by a valve in its bottom, which opens inwards to the receiver or close vessel. The piston is also furnished with a valve, which opens outward; so that when the piston is thrust down, the air in the barrel opens the valve of the piston, and makes its escape. After the piston has been forced down to the bottom, it is drawn up when its valve shuts, and that in the bottom opens, and admits the air from the receiver. This operation is repeated, and the air in the receiver becomes continually rarer.

The air-pump was invented by **Otto Guericke**, a gentleman of Magdeburg in Germany, about the year 1654, and marks a great era in the history of the physical sciences. It has received great improvements. **Cavallo**, vol. ii. p. 465, &c. **Hutton's** *Mathematical Dictionary*, article *Air-pump*.

345. When the number of strokes of the piston increases in arithmetical progression, the quantity of air remaining in the receiver, and also its density, form a series of terms decreasing in geometrical progression; so that if R be the capacity of the receiver, B of the barrel, D the density of the air in the receiver, after n

strokes of the piston, the density of the external air being 1; $D = \left(\frac{B}{B+R}\right)^n$.

Hence $\log D = n \log \frac{B}{B+R}$; from which D may be found when n is given, or n when D is given.

It is evident that the exhaustion can never be complete, as no more than a certain proportion of the air that remains can be taken out by the next stroke of the piston. There is even a limit which it cannot exceed in the ordinary air-pump; for the spring of the remaining air, being the power which opens the valves, when it becomes too weak to overcome their stiffness and their weight, the exhaustion necessarily ceases. In air-pumps of the most improved construction, the valves are opened by a mechanical contrivance, so that this limit to the perfection of the instrument is removed.

◆

346. When the end of a cylindric barrel constructed as above, is either immersed in water, or made to communicate with a reservoir of that fluid, on the piston being drawn up, and the air in the barrel rarefied, the water will ascend on account of the pressure of the external air. By repeated strokes of the piston, it may be made to rise so high, as to pass through the valve in the

piston, and a quantity of it is then lifted up every time the piston is raised.

This machine is called the *Sucking Pump*, and the piston is called the *Sucker*.

The valve of the piston, when lowest, must be nearer to the surface of the water in the reservoir than 33 feet, otherwise the water can never rise above it.

347. The piston of the sucking pump when it is raised, sustains the weight of a column of water, of a base equal to the section of the piston, and a height equal to that at which the water in the pump stands above the surface of the reservoir.

This is the measure of that part of the pressure of the external air that is not counteracted by the elasticity of the air under the piston.

To work the pump, a force must, therefore, be employed able to overcome the weight of this column, and of the piston, together with the friction of the piston against the sides of the pump.

348. In a sucking pump, the height of the lower, or fixed valve, above the surface of the water $= a$, the length of the stroke of the piston $= b$, and the height of a column of water in equilibrium

um with the pressure of the atmosphere $= h$; the height to which the water is raised by the first stroke, is $\frac{a+b+h-\sqrt{(a+b+h)^2-4bh}}{2}$.

BOSSUT, *Hydrodyn.* vol. I. p. 101.

349. The same notation being retained, and c being put for $a+b$, or the greatest height to which the piston ascends, b must be greater than $\frac{c^2}{4h}$, otherwise the water will not rise above the piston.

The water will cease to rise, when the air in the barrel of the pump, which is always rarer the higher the water is raised, becomes so rare, that when the piston is thrust down, it is not so much condensed as, by its elasticity, to open the valve in the piston, or, which is the same, is not so dense as the external air. This leads to the limit just assigned.

Besides the pump just described, there is one with a solid piston, without any valve, by the raising of which, the water is drawn up through the valve in the bottom of the pump, and on thrusting down the piston, the water is forced through a pipe in the side, having a valve that opens outwards, by which means it can be raised to any height if a sufficient force be applied. This is called the *Forcing Pump*.

The Lifting Pump is in principle no way different from the Sucking Pump.

350. The discharge of water by pumps, though naturally intermitting, may be rendered continual by the elasticity of the air condensed in what is called an *air-vessel*. This contrivance is represented in fig. 23.

See Desagulier, *Course of Experimental Philosophy*, vol. II. p. 152, 3d edition. Muschenbroeck, § 124, &c. Bossut, *Hydrodyn.* vol. I. § 76, &c.

Steam Engine.

351. Let a cylinder ABCD (fig. 24.), placed vertically, have a piston working in it, the rod of which is fixed to the end of a beam GH, turning on the axis O, and loaded at its other extremity by a weight W, and when the machine is at rest, let the beam incline toward the side where the weight is. Then if a vacuum be any how made in the part of the cylinder under the piston, the whole pressure of the atmosphere will tend to depress the piston, and will raise the weight W, if its mo-

mentum be not greater than that of a column of water 33 feet high, having for its base the section of the cylinder, and acting by the lever GH.

In this way, an air-pump might be made to raise water to any height whatever. But the production of the vacuum by the mechanical means of valves and pistons, from its difficulty and expence, could not be reduced to practice. For this reason, recourse is had to steam, which, from its facility of being generated and destroyed, affords the means of producing a vacuum over a great extent, and of renewing it at pleasure.

Suppose, therefore, the steam of water from a boiler to be introduced through a valve into the cylinder, and to expel the air; and when this is done, the valve to be shut, and another valve opened, by which a jet of cold water is injected into the cylinder, the steam will be quickly condensed; a vacuum will be produced, and the piston will be forced down by the pressure of the atmosphere.

The introduction of the steam at the bottom of the cylinder will again elevate the piston, and the reproduction of the vacuum will cause the depression of it; so that the end of the lever to which W is attached, may be employed to raise any weight, or to pump up any quantity of water within certain limits. This is the Steam Engine in its simplest state.

352. If b be the length of the stroke of the piston, r the radius of the cylinder, and h the height of the column of water which balances the pressure of the atmosphere, $\pi r^2 h$ is the moving force, and its momentum is $\pi r^2 h \times b$. If the radius of the piston of the pump worked by the beam be r', d the depth of the water, and h' the height to which it is raised at one stroke, $\pi r'^2 d h'$ will be the momentum of the resistance: then, in order that the machine may work, $r^2 b h > r'^2 d h'$.

To a complete theory of the steam engine, much more is necessary than the knowledge, that $r^2 b h$ is greater than $r'^2 d h'$; for the rate of working depends on that excess, and must be determined as in the problem, N° 182. The weight of the pump rods must be included, and also the effect of friction.

It is here supposed, that a perfect vacuum is produced in the cylinder by the jet of cold water. This, however, is not the fact, for by the alternations of the heat and cold to which the cylinder is exposed, it can neither acquire the heat necessary to the full elasticity of the steam, nor the cold necessary for its complete condensation. On this account, the effect of the machine falls much under the computation.

353. When the cylinder is full of steam, if a valve be opened, by which the steam is allowed to

escape into another vessel, where a jet of cold water is introduced, the condensation is much more complete, and the heat of the cylinder being preserved, the steam possesses its full elasticity.

This improvement was made by Mr Watt, and completely changed the character of the steam-engine. In the old engines, the power was reduced nearly to half its real value, so that the moving force, instead of amounting to 14 lb. on every square inch of the area of the piston, was reduced to little more than seven. In Mr Watt's engines, the moving force is not less than 12 lbs. on the square inch.

354. A farther improvement has been made on this engine, by injecting the steam into the cylinder, alternately above and below the piston, so that the whole motion is produced by the elasticity of steam, and has no dependence on the weight of the atmosphere.

This improvement is also due to Mr Watt, and could not have been made without the previous contrivance of condensing the steam in a separate vessel. It is particularly accommodated to the production of a rotatory motion by means of a steam engine.

In the double-stroke engine, the piston rods require to be forced down as well as to be drawn up, in the same vertical line. The method by which Mr Watt has accomplished this depends on a geometrical theorem.

355. If

354. If the ends of a line given in magnitude, describe circles given in position, and having their convexities turned opposite ways, a point may be found in it, which will describe on each side of the line joining the centres of the circles, an arch of a curve of contrary flexure, not differing sensibly from a straight line.

The point to be found divides the given line in the ratio of the radii of the circles described by its extremities.

A rectilineal vertical motion is also produced by another construction. Two of the adjacent angles of a parallelogram, are made to describe concentric circles, so that the side between them passes through their centre, and one of the remaining angles another circle, having its convexity opposed to that of the two former; then the third angle of the parallelogram describes a line that differs insensibly from a straight line.

Prony treats of this motion, *Arch. Hydraulique*, tom. II. § 1478. &c.; but has not given a complete theory of it. He says, on the authority of Adams, (*Geometrical and Graphical Essays*) that the contrivance was suggested to Mr Watt by one of the instruments for describing curves, invented by Suardi. See *Nuovi Instrumenti per la descrezione di diverse Curve*, &c. We have looked into this work with considerable attention, but have not been able to find any thing that has much affinity to the motions just described.

355. In steam engines of similar construction, the effects are nearly as the quantity of fuel consumed.

In this estimate time is necessarily involved, as, in order to derive the greatest advantage from a given quantity of fuel, the combustion must neither be too quick nor too slow.

It is computed, that an engine of the best construction, will raise 20,000 cubic feet of water to the height of twenty-four feet for every hundred-weight of good pit-coal. An engine with a cylinder of thirty-one inches diameter, and making twelve strokes in a minute, will do the work of forty horses, and will burn 11,000 lb. of the best Staffordshire coal in a day. See *Encyclopædia Britannica*, article *Steam*, p. 769. PRONY, *Arch. Hyd.* § 1499. vol. II.

Motion produced by Gunpowder.

356. When gunpowder is fired, a permanently elastic fluid is generated, which being very dense, and much heated, begins to expand with a force at least 1000 times greater than that of air under the ordinary pressure of the atmosphere.

ROBIN's *Tracts*, *New Principles of Gunnery*, Prop. VI. HUTTON's *Mathematical Dictionary*, art. *Gunpowder*.

In the acceleration of a ball by the fluid thus generated, three circumstances must be attended to.

357. 1. The elasticity is inversely as the space which the fluid occupies, and, therefore, as it forces the ball out of the gun, it continually diminishes.

2. The elasticity would diminish in this ratio, even if the temperature remained the same; but it must diminish in a much greater ratio, both from the dispersion of heat, and the absorption of it by the fluid itself, during its rarefaction.

3. The air propels the ball, by following it, and acts with a force that is, *cæteris paribus*, proportional to the excess of its velocity above the velocity of the ball. The greater the velocity that the ball has acquired, the less, therefore, is its momentary acceleration.

The effect of the elastic fluid must, for these reasons, decrease much faster than the space it occupies increases; and a formula expressing the law of acceleration, as depending on all these causes, more especially on the latter, might be expected to be very complex. Nevertheless, it appears from Dr Hutton's experiments, that the velocity with which a ball actually issues from the mouth of a cannon, other things

things being the same, is nearly as the square root of the weight of the gunpowder, or more generally, if v be the initial velocity of the shot, P the weight of powder, and B the weight of the ball, that $v = m\sqrt{\frac{P}{B}}$; m being a constant co-efficient, to be determined by experiment.

Philosophical Transactions, 1778. Also HUTTON's *Course of Mathematics*, vol. III. p. 270.

Now that $v = m\sqrt{\frac{P}{B}}$, is exactly the conclusion that would be deduced, from supposing the acceleration of the ball to depend simply on the expansion of the elastic fluid, without taking into account the diminution of the impulse, arising from the velocity which the ball has already acquired. This may be shewn from the *Principles of Dynamics*, § 96. That diminution, therefore, does not affect the ratios of the velocities of shot due to different charges, though it must no doubt affect the absolute quantities, and render them less than they would be otherwise.

The value of v given above, is only exact on the supposition, that the piece is long enough to allow the elastic fluid generated by the gunpowder to produce its full effect. The augmentation of the charge may so much lessen the space over which the fluid is to act, that the velocity of the ball

shall

shall be less from a great charge than from a smaller.

358. The Balistic Pendulum is an instrument invented by Mr ROBINS, for determining the velocity with which balls are projected by ordnance of different kinds. The ball is made to strike a heavy block of wood, suspended from a centre; the velocity is thus reduced to a quantity that is moderate, and easily admitting of being measured. From thence the original velocity is computed by help of the principles explained, Mechanics, Sect. 7. on the Rotation of Bodies, § 211.

For a particular description of this instrument, see ROBINS' *New Principles of Gunnery*, p. 84. Also EULER's *Notes on Robins' Gunnery*. The rule for computing the velocity of the ball from the vibration of the pendulum is there investigated, as it is likewise, HUTTON's *Tracts*, p. 115, &c. The initial velocity of shot estimated in this way, varies from 1600 to 2000 feet in a second. The latter is nearly as great as it is ever found.

The velocity with which the elastic fluid, disengaged from the gunpowder, expands itself, is probably more than double of this. Dr HUTTON estimates it at 5000 feet *per* second. *Course of Mathematics*, vol. III. p. 270.

359. The depth to which a ball penetrates into wood, earth, &c. is nearly as its weight multiplied into the square of its velocity.

Hence, since $v^2 = m^2 \times \frac{P}{B}$, $B v^2 = m^2 P$; so that the effect of a shot is nearly as the quantity of gunpowder.

From the Table given by Dr Hutton, *Course of Mathematics*, vol. III. p. 272. the value of m may be deduced nearly = 2000.

Bodies impelled by Currents in the Atmosphere, or by Winds.

360. The impulse of a stream of air striking with the velocity v, on a plane of the area a^2, inclined at an angle i to the direction of the stream, $= m \times a^2 v^2 \sin i^2$.

m is a quantity to be determined from experiment, and is constant, while the density of the fluid continues the same.

This proposition is the same that was formerly enunciated of non-elastic fluids, § 301.

361. The sails of a windmill are so disposed as to turn in a vertical plane round a horizontal axis, they themselves being all inclined to that plane toward the same side; when, therefore, the plane in which the sails turn, is placed at right angles to the direction of the wind, the force on each sail is resolved into two, one at right angles to its surface, and the other parallel to it. The latter has no effect, but a part of the former impels the sail, on the side toward which it makes an acute angle with the direction of the wind. Thus a circular or rotatory motion is given to the sails.

The sails are so constructed, as to have different inclinations to the plane of their motion at different distances from the axis, greatest nearer the centre, and least at their extremities. This is called the *weathering* of the sails, and is done in order to make the momentum of the wind nearly the same at all different distances from the centre of motion.

362. Supposing the sail of a windmill to be a plane, and the inclination of that plane to the axis or the direction of the wind to be i, the effect of the wind to turn the sail, in a plane at right angles to its axis, will be the greatest when $\cos i \times \sin i^2$ is a maximum, or $\cos i = \frac{1}{3}$.

This gives $i = 54^\circ 44'$, and therefore the inclination of the sail to the plane of its motion, or what is called the angle of *weather*, 35° 16′. This is true only when the sail is at rest, or just beginning to move. When the sail is in motion, and of course near the extremities of the sail, when it moves faster, the angle of weather must be less. **Maclaurin** has given a formula for this, *Fluxions*, vol. II. § 913, 914.

Maclaurin's theorem makes the *weather* to vary from 26° 34, at the point of the sail nearest the centre, to 9° at its extremity. **Smeaton** made some corrections on this, from experiment. See *Experimental Inquiry* concerning *Mills*, p. 45, and 46.

363. From Smeaton's experiments it appears, that a windmill works to the greatest advantage, when it is so constructed that the velocity of the sails, is to their velocity when they go round without any load, as a number between 6 and 7 is to 10; and also that the load, when the mill works in this manner, is to the load that would just keep it from moving, nearly as 8.5 to 10.

Experimental Inquiry, p. 49 & 50.

364. With different velocities of wind, the load that gives the maximum *effect*, varies nearly as

the square of the velocity of the wind, and the *effect* itself as the cube.

> *Ibid.* p. 52.
>
> **The *effect* is always measured by the product of the velocity of the load into its weight. The velocity of the load varies in the simple and direct ratio of the velocity of the wind.**

Resistance of the Air to Projectiles.

365. Though we should be led, as in hydraulics, to conclude that the resistance which air makes to moving bodies is as the square of their velocities, experiment appears to prove, especially when the velocity is great, that the resistance is partly proportional to the square, and partly to the simple power of the velocity.

> **The resistance to the same ball, its velocity being v, is $m v^2 + n v$; where m and n are given co-efficients. HUTTON'S *Course of Mathematics*, vol. III. p. 278.**
>
> **In this formula $m = .00002576 = \frac{1}{38825}$, and $n = -.00388 = -\frac{1}{258}$, the diameter of the ball being 1.965 inches.**

Hence for any ball of which the diameter is d, $m = .00000666$, and $n = -.001$, that is,

$$m = \frac{2}{3000000}, \text{ and } n = -\frac{1}{1000};$$

and the resistance in avoirdupois pounds, or

$$r = \frac{d^2}{1000}\left(\frac{2v^2}{3000} - v\right).$$

For this theorem, so useful in gunnery, and so well accommodated to practice, we are indebted to Dr Hutton of Woolwich, *ubi supra*.

A ball is often resisted by a force that is many times its own weight. An iron ball 3 lbs. weight has its diameter = 2.78 inches, and when it is thrown with a velocity of 1800 feet, it is resisted by a force equal to 176 lbs., more than 58 times its own weight.

366. Supposing the air to resist according to the law just assigned; the height, to which a ball of the weight w, and the diameter d, will ascend, when projected perpendicularly upwards with any velocity c, will be,

$$\frac{w}{2gd^2m} \times \text{hyp. log.} \frac{c^2 - \frac{n}{m}c + \frac{w}{md^2}}{\frac{w}{md^2}}.$$

If this expression be reduced, by substituting for w its value $.5236\,d^3$, and using the common logarithms, $h = 760\,d \times \log\left(\frac{c^2 - 150\,c}{21090\,d} + 1\right)$.

It may be shewn by this, that a ball 1.05 lb., discharged with a velocity of 2000 feet, will ascend only to the height of 2920 feet, or little more than half a mile, whereas *in vacuo* it would have ascended to the height of eleven miles and three-fourths. Hutton, *ibid.* p. 284.

367. If a body descending in the atmosphere, has acquired such a velocity, that the resistance is equal to its weight, the accelerating and retarding forces being equal, its motion will become uniform.

Hence, since $.5236\,d^3$ is the weight of an iron ball of the diameter d, if we make

$$.5236\,d^3 = \frac{d}{1000}\left(\frac{2\,v^2}{3000} - v\right),$$ (see article 7.), or

$$523.6\,d = \frac{2\,v^2}{3000} - v,$$ we have a quadratic equation,

from the solution of which v may be found. The velocity thus found is called the *terminal velocity* of the falling body, or of the projectile. For an iron ball of 1 lb., the terminal velocity is 244 feet; for one of 42 lb., it is 456. *Vid.* Table, Hutton's *Course*, vol. III. p. 291.

In strictness this velocity is not acquired till after an infinite time, and a descent infinitely long: the time of acquiring a velocity differing from it, by any quantity, however small, is finite, and can easily be assigned.

368. The great problem in gunnery, viz. having given the weight, the magnitude, the direction, and the velocity of a projectile, to determine its path through the air, supposing the law of resistance to be known, is very difficult, and the solutions of it hitherto given have not led to results easily applicable to practice.

An approximation has been given by Newton, and complete solutions by Euler and Le Gendre. *Prin. Math.* lib. II. Prop. 10. Euler's *Remarks* on Robins, Brown's translation, p. 322., &c. Le Gendre's solution is in Francœur's *Mechanique*, p. 196., &c. 3d edit.

Dr Hutton of Woolwich is in possession of many valuable materials relative to this problem, furnished by his own experiments, and there is reason to expect a more useful solution than has yet been given, from one who unites profound mathematical knowledge with great practical skill.

D'Antoni has proposed a method of determining by experiment an indefinite number of points in the path of the same projectile.

If a series of stations, one above another, be taken on the declivity of a hill, at the bottom of which there is stretched out a plain of considerable extent; and if the same piece of artillery be carried successively to these stations, and fired at the same angle, and in the same circumstances in all respects, and the points marked where the shot in all these cases strikes the plain, it is evident, that all these are points in the curve described by the same shot.

The same might be done still more easily, by removing the gun from one distance to another in the plain, and firing against the side of the hill. If the points from which the shot were fired, and those which they struck, were all marked on an accurate profile of the ground, they would enable us to determine as many points in the path of the projectile as there were experiments made. It is unnecessary to observe, that at each station a great number of shot might be fired at different elevations, so that many curves might be determined from the same set of experiments; and that at each particular elevation many shots ought to be fired, and a mean taken among the points where the balls struck.

369. The ranges of the same ball, with the same elevation, but different charges, are nearly as the square-roots of the initial velocities; and the times of flight are nearly as the ranges.

HUTTON'S *Mathematical Dictionary*, article *Gunnery*, p. 567. Both these propositions are *empirical*, or deduced solely from experiment.

Beside the works already quoted, see the article *Resistance*, by Professor ROBISON, in the *Encyclopædia Britannica*.

SECT. II.

AIR AS THE VEHICLE OF SOUND.

Vibration of Sonorous Bodies.

370. The bodies that we consider as *sonorous*, or as causes of the sensation of sound, at the time when they produce that sensation, are observed to be in a state of tremor or vibration.

This is exemplified in a bell, and in a drinking-glass when sound is produced by rubbing its edge with a wet finger, in a musical string, &c. The latter affords the simplest case of this fact, and that in which the laws of the vibration are most easily investigated.

371. If a musical string stretched between two fixed points, be struck any where between those

points, so as to be forced out of the straight line by a small quantity, it will vibrate backwards and forwards on each side of that line, and the curves into which it will successively pass in the course of these vibrations, will have their curvature at every point proportional to the distance from the straight line joining the fixed points; the accelerating force at each point will also be proportional to that distance, and the great and small vibrations will be performed in the same time.

It is usual to reckon the vibrations of a string differently from those of a pendulum; the passage from the highest point on one side, to the highest on the other, is reckoned *a vibration* of a pendulum. The passage from the farthest distance on one side to the farthest on the other, and back again to its first position, is accounted a *vibration* of a musical string. It is properly a double vibration.

372. The figures which a musical string assumes in its vibrations, constitute a series of elastic curves, or elongated cycloids.

The construction of these curves is easy. See the Notes in LE SEUR, and JACQUIER's Commentary on the *Principia*, lib. II. prop. XLII. the Note 301. Also SMITH's *Harmonics*.

373. If L be the length of a musical string of uniform thickness, P its weight, S the weight by which it is stretched, or the measure of its tension, t the time of a double vibration,

$$t = 2\sqrt{\frac{LP}{gS}}.$$

t, as usual, is measured in seconds, and g is the velocity acquired in 1″ by a falling body, expressed in the same measure with L. The most convenient unit in this case is an inch; so that $g = 386$.

The value of t may be made more convenient, by supposing w to denote the weight of an inch of the string; so that $P = wL$, then $t = \frac{2L\sqrt{w}}{\sqrt{gS}}$.

If N be the number of vibrations in a given time, for instance in one second, $N = \frac{\sqrt{gS}}{2L\sqrt{w}}$.

If the lengths and weights of two chords are the same, their times of vibration will be inversely as the square roots of the forces by which they are stretched; and the number of vibrations which they perform in the same time, directly as those square roots.

So also, the tension and the weight remaining the same, the celerity of the vibration is inversely as the square root of the length; or, the tension and weight *per* inch remaining the same, the celerity of vibration is inversely as the length.

The problem of the musical chord was first resolved, and the preceding theorems investigated, by Brook Taylor, in his *Methodus incrementorum*, prop. 21. 23., &c. His solution, though very ingenious, and extending to most of the cases that occur in nature, was not general, nor complete, when mathematically considered. The first complete solution was given by D'Alembert, in the *Berlin Memoirs* for 1747, and led him at the same time to discoveries which form a great era in the history of the Differential and Integral Calculus. *Mém. de Berlin*, 1747, p. 214., &c.

374. When the sounds of different musical strings are compared, a certain difference between them is perceived by the ear, which is called *difference of tone*; and this difference is also expressed by saying, that the one sound is *graver*, and the other more *acute*; the variations of tone are found to have a constant or fixed relation to the comparative celerities of vibration.

As the string performs more vibrations in a given time, the sound it yields becomes more acute; and as it vibrates more slowly, the sound is graver. This is easily brought to the test of experiment. The strings that vibrate faster, either from the greater tension, or their smaller length and weight, invariably produce sounds that are more acute, &c.

375. If eight strings be such, that the numbers of the vibrations they perform in a given time are as the numbers 24, 27, 30, 32, 36, 40, 45, 48, the sounds of the first seven will be perceived as increasing in acuteness one above another, from the first to the last, and will yield the notes from the combinations of which all musical effects are produced.

The tone is not affected by the extent of the vibrations, but merely by their time. The loudness of the sound is supposed to depend on the greater extent of the vibrations. Noise and discordant sounds arise from a want of isochronism of vibration. When the vibrations are isochronous, or all performed in the same time, the sounds are always musical.

The last of the strings will sound what is called the *octave* above the first, and the same series may be repeated again between the number 48 and its double 96, and each note will be an octave above its corresponding note in the first interval; the numbers of vibrations will be 54, 60, 64, 72, 80, 90, 96; and it is evident that this series may be continued either up or down without limit.

The musical scale thus formed, is called the *Diatonic Scale*.

The pleasure derived from the successive or simultaneous perception of the sounds of this series, appears to be an ultimate fact that can be no farther analysed, but must be referred to the original constitution of the mind. The selection of the combinations of these notes, capable of affording high degrees of pleasure, is the object of the musical art.

377. All other sonorous bodies, at the time they emit sound, vibrate in like manner, but according to laws less simple. In wind instruments, the sounding and vibrating body is the air itself.

The number of vibrations performed in a given time, by any sonorous body, may be determined by comparing its sound with the note which is sounded by a musical string of a given length, weight and tension. The ear is sufficient to decide what string in a harpsichord is in unison with the given sound. The number of vibrations performed in a given time by the former, is equal to the number of those performed in the same time by the latter.

378. All musical sounds are computed to be contained within ten octaves; so that the number of vibrations in a given time that yields the gravest note, is to that which yields the most acute, as 1 to 2^{10}, or as 1 to 1024.

Propagation

Propagation of Vibrations through the Air.

379. The vibrations of sonorous bodies are communicated to the air, and by the impression thus made on the ear, excite the sensation of sound.

That air is necessary to the production of sound, is evident from including a bell in a receiver, exhausting the air, and making the clapper strike on the bell: the sound is hardly audible.

380. It is not every kind of vibratory motion produced in the air that is the cause of sound; a musical string may vibrate, but if it is touched by a bit of cloth, or any soft body, no sound is heard. The vibrations in the air that produce sound must be communicated by some elastic substance.

Sound is produced by the explosion of gunpowder, that is, by the sudden extrication of a fluid mass, dense, and highly elastic. It is produced also by the sudden rushing in of air to supply a vacuum. The crack of a whip appears to be an example of this last.

381. Though the vibrations of the air that produce the sensation of sound, are no doubt the al-

ternate condensations and rarefactions of that fluid, in consequence of which the particles go and return, or oscillate backwards and forwards for some time, even though there is no renewal of the impulse of the sonorous body, yet every such oscillation acts only once, or by a single impulse on the ear. Were it otherwise, sound would be always something inarticulate and ill defined.

382. The velocity with which vibrations are propagated through the air, is the same that a heavy body would acquire by falling through half the height of the *homogeneous* atmosphere, or that which the atmosphere would be reduced to, if it were everywhere of the same density, and the same temperature with the air at the surface of the earth.

The height of this homogeneous atmosphere has been formerly computed at 4343 fathoms, when the temperature is that of freezing. If this height be called H, then v, the velocity of the aerial vibrations, $= \sqrt{2gH}$. Hence $v = 1057$, which is too small, as it is found by experiment to be 1142 feet *per* second.

The nature of the vibrations of an elastic fluid, was first considered by NEWTON, and their velocity defined, as in the preceding proposition, by comparing them with the vibrations of a pendulum. *Princip.*

Princip. Math. lib. ii. prop. 47. & 49. The soundness of this reasoning, however, was questioned by **Euler**; and it was afterwards shewn by **Cramer**, that the same argument might be used to prove conclusions very different from that at which Newton had actually arrived. *Commentary on the Principia*, lib. ii. vol. ii. p. 364. La Grange afterwards gave an accurate solution of the problem, by which, though he pointed out the error of Newton's reasoning, he confirmed the truth of his conclusion.

383. The velocity of the pulses propagated in an elastic fluid, are as the square root of the elasticity divided by the density of the fluid.

Prin. Math. ibid. prop. 48.

The velocity of sound, computed as in the last article, comes out less than by actual experiment. La Place has suggested a very probable explanation of this, viz. that the condensation of the *undulæ*, which must take place in these vibrations, produces a degree of sensible heat, by which the elasticity is increased, or, to speak more correctly, the density diminished, while the elasticity remains the same. The heat required for that effect has been shewn by Biot to be within the probable limits which analogy, in the absence of direct experiment, would lead us to assign. Haüy, *Leçons de Phys*, § 478.

384. There is perhaps, in the economy of nature, no contrivance more wonderful, than that by which things apparently so little susceptible of precision as the impulses communicated to an elastic fluid, become the means of conveying to the mind such a multitude of distinct impressions as it receives through the ear; the finest modulations of harmony, and the nicest distinctions of articulate language.

385. An Echo is a repetition of sounds produced by the reflection of the aërial pulses that convey sound to the ear.

A wall, a rock, a grove of trees, may be so placed, as to cause an echo. That an echo may return one syllable as soon as it is pronounced, the reflecting surface should be 80 or 90 feet distant; for a dissyllabic echo, 170 feet, &c. The sound, whether direct or reflected, is supposed to proceed at the rate of 1142 feet in a second.

386. The Speaking Trumpet is an instrument intended to transmit the sound of the voice in some particular direction, to a greater distance than it would otherwise reach.

The best form of a speaking-trumpet, is found to be that of a hollow cone, with a mouth-piece at the narrow

narrow end, to receive the lips, and confine the voice of the speaker.

The beat of a watch may be heard to twice the distance through a speaking-trumpet, that it can be heard at without one. It is said, to be the same in this experiment whether the trumpet is cylindrical or conical. See the *Edinburgh Encyclopædia*, article *Acoustics*, Part ii. Sect. ii. This instrument seems to produce its effect, by preventing the pulses, as they are generated, from diffusing themselves all round, and subjecting them longer to an impulse in one direction.

SECT. III.

AIR, AS THE VEHICLE OF HEAT AND MOISTURE.

387. THE Earth and the Atmosphere, taken generally, receive at all times nearly the same quantity of heat and light from the sun.

As the whole of one side of the earth is constantly turned to the sun, the only difference in the quantity of heat and light which the earth receives in a given time, must arise from the changes which take place in its distance from the sun at different seasons of the year. In consequence of these changes, the quantities of light and heat received by the earth are not proportional to the times, but to the angles described by the earth round the sun in those times. These variations, however, are but inconsiderable, and as they are annual, they do not produce any inequality in the whole heat or light of one year compared with those of another.

388. Though the earth is thus receiving heat continually, and nearly at the same rate, its average temperature appears to remain invariable; as much heat as comes from the solar rays, flying off continually into the fields of vacuity, or of ether.

389. While the general temperature of the earth remains invariable, the distribution of heat over its surface is extremely unequal, being different in different places, and in the same place subject to variations, both regular and irregular.

390. The causes that determine the distribution of heat over the earth's surface, are either the direct influence of the solar rays, or the communication of heat by the air from one part of the earth's surface to another.

391. The first of these depends on the latitude of the place, or its distance from the equator, by which the intensity of the heat and light from the sun, and also the length of the day, are determined for different seasons of the year.

> The intensity of the sun's rays, when they strike on any plane, is as the quantity that falls on a given space, or as the sine of the sun's elevation above the plane. The nearer the sun is to the zenith of any place, at a given moment, the greater the intensity of heat produced by his rays.
>
> The heat for an entire day, depends also on the length of the day, and as the day is longer where the distance from the zenith is greater, the inequality in the distribution of heat arising from the one of these causes, compensates that proceeding from the other, and brings their combined effects nearer to an equality than might be imagined. **Fontana** has shewn, that the heat of the day of the summer solstice at Pavia, is greater than the heat of the same day at Petersburgh only in the ratio of 63° to 62°, though the latitude of the former be 45° 11′, and of the latter 59° 56′.
>
> The same author finds, that when the sun's declination exceeds 18°, or from about the 10th of May to the 30th of July, the heat in twenty-four hours proceeding from the sun's rays is greater at the north pole than at the equator. **Greg. Fontana**, *Disquisitiones Physico-Mathematicæ*, 1ma & 2da.

The distribution of heat, therefore, if only the direct influence of the sun were to act, would be very different from that which takes place in nature.

The intensity of the solar heat is less in low elevations than is here supposed, on account of the rays coming through a larger mass of air, and of air more loaded with vapour, so that a great quantity of them is intercepted. BOUGUER, *Traité d'Optique; sur la Gradation de la Lumiere,* Liv. iii. sect. iv.

392. The effects of the direct influence of the sun, are greatly modified by the transportation of the temperature of one region into another, in consequence of that disturbance in the equilibrium of the atmosphere which the action of those rays necessarily produces.

In order that there may be an equilibrium in a fluid like the air, every stratum of air that is level or horizontal all round, ought to be of the same density. It ought, therefore, also to be everywhere of the same temperature, which not being the case, the constant motion of the air is the necessary consequence of heat being unequally distributed. The columns of air that are lighter, are displaced by those that are heavier, and hence a general tendency in the air to move from the poles toward the equator. This general tendency, which is calculated to moderate the extremes of temperature, is also greatly modified by local circumstances.

393. The sea is preserved of a moderate temperature by the statical principle that makes the heavier columns of a fluid displace the lighter. A more uniform temperature is thus given to the sea, which communicates itself to the air incumbent on it, and to that on every side.

394. Conversely, the effect of great and unbroken continents, is favourable to the extremes of heat or of cold.

The constitution of the surface may tend to increase and sometimes to diminish this effect. High mountains especially, if covered with snow, may enforce the rigour of a cold climate, or temper the heats of a warm one.

Forests tend to increase the cold, by preventing the sun's rays from striking on the ground. Evaporation produces cold; and marshes and lakes are therefore favourable to the severity of the weather. The congelation of water produces heat, and moderates the cold; the melting of ice, on the other hand, increases the capacity for heat, and so produces cold.

395. Height above the level of the sea, causes a diminution of heat at the constant rate of 1° for 270 feet nearly, when not far from the surface of the earth.

It has already been remarked, that this decrease seems to be somewhat slower as we ascend, but not very considerably, as far as our observations have extended.

396. The combination of these causes gives to every place a mean temperature, which remains always nearly the same, and which decreases from the equator to either pole, according to a law that has been determined by observation.

397. Let t be the mean temperature of any parallel of which the latitude is L, M the mean temperature of the parallel of 45°, and $M + E$ the mean temperature of the equator then is ;

$$t = M + E \cos 2L.$$

In this formula, $M = 58°$, and $E = 27°$. When $2L > 90$, $\cos 2L$ is negative.

This theorem was first given by Mayer, *Opera inedita*, vol. i. p. 4., &c. Also Kirwan, *Estimate of the temperature of different Latitudes*, p. 18.

From this a geometrical construction, for finding the mean temperature, may be readily deduced.

In the line AC (fig. 25.), divided into equal parts, numbered from A, so as to represent the scale of a thermometer, let $AC = 85$, and $AB = 58$. From the centre

centre B, with the distance BC or 27, describe the semicircle CA; take the arch CG equal to the double latitude of any parallel; and from G draw GO perpendicular to AC; then is AO the mean temperature of that parallel, according to FAHRENHEIT's scale.

The mean temperatures thus found, agree very well with observation. Springs, in which the water does not considerably change its heat from one season of the year to another, afford an expeditious and accurate way of ascertaining the mean temperature.

398. If the place is at any height H above the level of the sea, $t = M - \frac{H}{270} + E \cos 2 L$.

H is understood to be expressed in English feet.

399. On ascending into the atmosphere, at a certain height in every latitude, a point is found where it always freezes, or where it freezes more than it thaws, so that the mean temperature is below 32°. The curve joining all these points, from the equator to the pole, is called the *line of perpetual congelation*. The equation to it will be found, by making $32 = M - \frac{H}{n} + E \cos 2 L$.

This line at the equator is elevated 15577 feet above the level of the sea; so that for determining n we have $32 = 58 - \frac{15577}{n} + 27$, or $n = 294$.

Thus, $H = 7642 + 7933 \cos 2L$, and this seems nearly to agree with actual observation.

Professor LESLIE has given a different equation, founded on the law of the diminution of heat mentioned at § 339. In the table calculated from it, heights come out rather below what observation requires. *Elements of Geometry*, 2d edition, p. 495.

400. The temperature of the latter end of April is observed, at least in the temperate zone, to be nearly the mean temperature of the year. From that time the heat increases, and is at its maximum about the 21st of July; it goes on decreasing from that time till it come to the mean in the end of October, and it passes from thence to the greatest cold about the 21st of January. All these vicissitudes may be nearly represented by the formula in which G is the mean temperature for the given place, F a constant co-efficient to be found by observation, λ the mean longitude of the sun computed from the first of Aries, for any day of the year, the mean temperature of which is y. Thus, the times of the mean and the extremes of temperature being taken by about one month la-

ter than the corresponding mean and extremes of the sun's elevation,

$$y = G + F \sin(\lambda - 30).$$

This supposes that the mean heat takes place when the sun enters Taurus, which is nearly true. In this latitude, we may suppose $F = 15°$.

401. From this formula, compared with the two former, one theorem may be deduced, including the effects of latitude, elevation above the surface, and the season of the year, viz., the mean temperature of any day under any parallel, and with any elevation,

$$y = M - \frac{H}{270} + E \cos 2L + F \sin(\lambda - 30°).$$

402. The formula for the mean temperature, as laid down above, would probably hold over the whole globe, if it were everywhere covered by the ocean; but it agrees strictly only with the Atlantic Ocean, and the western part of the Old Continent.

The editor of MAYER's Posthumous Works observed, that his theorem applied accurately only to the portion of the globe contained between the parallels of Stockholm and of the Cape of Good Hope, and between the meridians of Stockholm and Mexico. This does not seem quite exact; no part of North America having its mean temperature the same with that of places of the same latitude in Europe. It would require at least 10° to be taken from M, to adapt the formula of § 397. to the New Continent. See Kirwan's *Estimate of the temperature of different Latitudes*, p. 15.

403. As we go eastward from the shores of the Atlantic, the mean temperature of any parallel becomes lower, at a rate that may perhaps, for the north part of the temperate zone, be estimated at a degree for 150 miles.

At St Petersburgh, lat. 59° 56′, about 750 miles from what may be accounted the shores of the Atlantic, the temperature is 5° 5′ below the standard. The medium temperature of January is no more than 10°. By computation from the formula above, it ought to be greater than 32°. The winter lasts from October to April, and the cold is sometimes as great as the freezing point of mercury, or — 39°. From a mean of several years, the mean of the winter cold is — 25°. Kirwan, *ibid.* p. 61.

It was at Krasnojark, lat. 56° 30′, long. 93° E. that mercury was first known to freeze by natural cold.

If we were to begin where any parallel intersects the shore of the Atlantic, and draw on the map a line along which the mean temperature should be constantly the same as at the first-mentioned point, it would incline greatly to the south. The point, for instance, in the meridian of Petersburgh, which has the same temperature with the standard belonging to the parallel of that city, is about 5° south of it, or in the latitude of 54° 30′ nearly.

At Irkutz, latitude 52° 15′, longitude 105° east, the mean temperature from October to April has been known to be as low as —6°.8, a temperature which for severity and duration exceeds any thing that has yet been observed elsewhere.

404. This increase of the severity of the winter, and the consequent diminution of the mean temperature, on going eastward, holds in all the latitudes north of the parallel of 30°; but the diminution is slower as we approach that parallel; to the south of 30° the mean heat increases on retiring from the ocean.

This diminution takes place all the way to the shores of the Pacific, or very near them. The climate of Pekin is vastly more severe than that of the same parallel (39° 54′) in Europe.

405. In the New Continent, also, at least in the part of it to the north of the Tropic of Cancer,

the mean temperature is much below the standard, and the severity of the winter much greater than in the same latitudes in Europe.

At Prince of Wales' Fort, Hudson's Bay, lat. 59°, long. 92° west, the mean temperature is 20° under the standard; at Nain in Labrador 16°; at Cambridge in New England (lat. 42° 25',) 10 degrees. Mercury has been supposed to be frozen by the natural cold as far south as Quebec, lat. 47°.

A very low mean temperature, and extreme cold in winter, are characteristic of the climate of North America.

406. In the higher latitudes of the Southern Hemisphere, the temperature is lower than in the same latitudes of the Northern Hemisphere.

Forster describes a small island on the coast of South Georgia, latitude 54° south, which, in the middle of summer, was covered almost entirely with frozen snow, to the depth of several fathoms.

The South Pole is surrounded to the distance of 18 or 19 degrees with a barrier of solid ice, through which even the skill and intrepidity of Captain Cook could not force a passage.

It is known also, that detached masses of ice float down in that hemisphere as low as the latitude of 40^d. The cause of this phenomenon is by no means sufficiently understood.

407. The varieties of temperature to be met with on the surface of the earth, appear to be confined between the limits of 100° and —40°.

No degree of cold much below —40° has ever been observed in nature, even when thermometers have been employed containing a fluid not liable to congelation. Sir CHARLES BLAGDEN *on the Congelation of Mercury*, p. 61.

The heat of 100° is rare, but a heat approaching to 90°, is found in the summer of most countries within the limits of the temperate zones.

There is hardly any climate, even in the frigid zone, where a temperature between 60° and 50° is not occasionally experienced.

The greatest heat is much less above the mean temperature than the greatest cold is below it. The mean temperature for the whole surface may be taken at 58^d; the greatest summer heat is only 42° above this; the greatest winter cold in 98° under it.

408. There is reason to think that the climates of Europe were more severe in ancient times than they are at present, and the change that has taken place may with great probability be ascribed to

the better and more extensive cultivation of the ground.

Cæsar says, that the vine could not be cultivated in Gaul on account of the severity of the winter. The rein-deer was then an inhabitant of the Pyrenees. The Tiber was sometimes frozen over, and the ground about Rome covered with snow for several weeks together. See Hume *on the Populousness of Ancient Nations*, *Essays*, vol. i. p. 451. edit. 1772. Daines Barrington, *Phil. Trans.* vol. lviii. (1768) p. 58., &c.

Cultivation may improve a climate, 1mo, By the draining of marshes, and lessening the evaporation, which is so great a cause of cold; 2do, By turning up the soil, and exposing it to the rays of the sun; 3tio, By thinning or cutting down forests, which by their shade, prevented the penetration of the sun's rays. The improvement which is continually taking place in the climate of North America, proves that the power of man extends to the phenomena, which, from the magnitude and variety of their causes, seemed most superior to his controul. At Guiana in South America, the rainy season has been shortened by the clearing of the country, and the warmth greatly increased. It thunders continually in the woods; rarely in the cultivated parts. Buffon, *Sup.* tom. ix. p. 346. octavo.

Wind.

Wind.

409. The principal cause of those currents of air to which we give the name of *Winds*, is the disturbance of the equilibrium of the atmosphere by the unequal distribution of heat.

In order that an equilibrium may take place in an elastic fluid, circumfused about a solid, to which it gravitates, every level stratum of the fluid, that is, every stratum, which, when continued round, cuts the directions of gravity everywhere at right angles, should be of the same density, and therefore of the same temperature. As this is not the case, the equilibrium of the atmosphere is inconsistent with the actual distribution of heat on the earth's surface.

The general tendency, in such circumstances, is for the heavier columns to displace the lighter, and for the air at the surface to move from the Poles toward the Equator. The only supply for the air thus constantly abstracted from the higher latitudes, must be produced by a counter-current in the upper regions of the atmosphere, carrying back the air from the Equator toward the Poles. The quantity of air transported by these opposite currents, is so nearly equal, that the average weight of the air, as measured by the barometer, is the same in all places of the earth.

If the surface of the earth were wholly covered with water, so that there was no part of it more disposed than another to obstruct the motion of the air, or that had a greater capacity than another, of acquiring or communicating heat, the air would probably circulate continually in this manner from the Poles to the Equator, and back again, without any irregularity whatsoever.

410. In consequence of the rotation of the earth on its axis, another motion is combined with that of the currents just described. The air, which is constantly moving from points where the earth's motion on its axis is slower, to those where it is quicker, cannot have precisely the same motion eastward with the part of the surface over which it is passing, and therefore must, relatively to that surface, describe a curve, having its convexity turned to the east. The two currents, therefore, from the opposite hemispheres, when they meet toward the middle of the earth, have each acquired an apparent motion westward; and as their opposite motions from south and north must destroy one another, nothing will remain but this motion, by which they will go on together, and form a wind blowing directly from the east.

This is the cause of the *Trade Wind*, which (with certain exceptions) blows continually between the

Tropics, or rather between 30° on the one side of the Equator, and 30° on the other.

The Trade Wind declines somewhat from due east toward the parallel to which the sun is vertical at different seasons of the year. As the sun approaches the southern tropic, the Trade Wind is directed somewhat to the south; and as he approaches the northern, somewhat to the north.

The cause usually assigned for the Trade Wind, is the constant motion toward the west of the spot to which the sun is vertical, and where of course the rarefaction is greatest. This, it is supposed, draws along with it the air from the east. This, however, is by no means satisfactory, and it seems certain, that if the Trade Wind were produced in this way, it must have great rapidity, in place of being a gentle breeze, at the rate of seven or eight miles an hour.

The opinion that the Trade Wind is produced by the air in its motion southward falling back toward the west, is mentioned, but rejected, by HALLEY. It has since been espoused by FRANKLIN and LA PLACE, and is, on the whole, less objectionable than any other.

The matter is here stated somewhat differently from what is done by those authors, particularly the effect of the currents from the opposite hemispheres, in determining the motion to be wholly from the east.

411. The attractions of the sun and moon have sometimes been considered as among the general causes of the winds. They have no doubt a tendency to produce in the atmosphere an undulation backwards and forwards, like the tides which they cause in the ocean. It does not appear, however, that they could produce any continued progressive motion of the air, similar to that of the Trade Winds. Their effects also, are too minute to be perceived, amid the action of so many more powerful causes.

D'ALEMBERT in his *Recherches sur les Causes Générales des Vents*, has treated of the forces of the sun and moon to produce currents in the atmosphere. His essay is more remarkable for the resource and ingenuity it displays in the management of the *calculus*, than for the physical conclusions to which it leads.

412. The superior current above described, restores the air carried from the higher latitudes to the lower, with such a degree of equality, that the average weight of the atmosphere, as measured by the barometer, is nearly the same in all climates. This restoration is, however, subject to great local and temporary irregularities, from the different

degrees of resistance that the air meets with in passing over the surface, and the different capacities of that surface for receiving and communicating heat.

The motion of the inferior and superior currents, may be seen exemplified on opening a door, between two apartments of different temperatures. The flame of a candle near the ground, will shew the stream that sets from the colder room to the warmer; near the top it will indicate a stream in the opposite direction.

As the average quantities of the air carried by these opposite currents are equal, the surface that separates them is probably not far different from that at which the barometer would stand at 15 inches, or half its medium height at the surface.

If we suppose the mean temperature is 32°, this elevation will be found =3010 fathoms, or 18,060 feet =3.425 miles, not so high as the summits of the Cordilleras.

The upper stream in each hemisphere being directed to one point, the pole of that hemisphere, there must arise a considerable condensation, and acceleration of the air, as the currents approach that point, and hence the causes of irregular winds must be increased on approaching the poles.

The general direction of the upper current must be to the westward, for the same reason that that of the lower was toward the east.

413. The Trade Wind itself is subject to certain irregularities. As the sun advances into the Northern Hemisphere, the Trade Wind becomes irregular; and about the middle of April, in all the tract between Africa and the peninsula of India, and much beyond it, changes from north-east to south-west, and continues to blow in that direction, till the sun returns to the Southern Hemisphere.

The cause of this change is difficult to be assigned. It seems probable, that by the sun's entry into the Northern Hemisphere, he communicates great heat to the sandy desarts of Africa, which lie to the west or south-west of the seas just mentioned. The great heat acquired by the sand of those desarts, produces a rarefaction in the columns of air incumbent on them, and consequently a tendency, in the columns that are near them, and more moderately heated, to flow in and displace the heated air. The air of the Atlantic is most likely to do this, and in passing over Nigritia, &c., to acquire a velocity that carries it on eastward through the Indian Ocean.

These periodical winds are called *Monsoons*.

The change, or the setting in of the monsoons, does not happen all at once. In some places the shifting of the wind is accompanied with calms; in others with variable winds, heavy rains, thunder, and violent storms.

414. The tract from the parallel of 30° to the Pole, in each hemisphere, is the region of variable winds; and their unsteadiness and violence seem to increase, the nearer they approach the Polar Circles.

The irregularity of winds proceeds from inequalities in the motion of the general currents above mentioned, and from a variety of local causes; also from the chemical changes that are carried on in the air, such as the solution and precipitation of moisture, and the action of the electric fluid on the different gases that compose the atmosphere.

415. Sudden and strong gales of wind appear almost always to arise from a diminution of the weight of the air in the tract where the wind prevails, and are accompanied, or preceded, by a fall of the barometer.

The sudden sinking of the barometer almost always indicates a gale of wind, though a gale that is sometimes at a considerable distance. When the barometer begins to rise, it is a symptom that the gale has reached its height; and though it may still continue to blow for a long time, it is usually with decreasing violence.

416. Notwithstanding these irregularities, there is in most countries a tendency to periodical

winds, more or less remarkable, according to the steadiness of the climate.

Even with us, where an insular situation, with a great Continent on one side, and a great Ocean on the other, unites all the causes of a variable climate, the East wind usually prevails in the spring, from the vernal equinox to the summer solstice, and beyond it; during the rest of the year, the Westerly winds prevail, though not without frequent incursions of the east, by which our most unpleasant weather is always produced.

The *Etesian*, or northerly wind, prevails very much in summer all over Europe. PLINY describes it as blowing regularly in Italy, for forty days after the summer solstice, lib. ii. cap. 47. It is part of the great current that carries the atmosphere of the higher latitudes down to the tropical regions.

417. The velocity of the wind varies from one that is hardly sensible, to one of 100 miles in an hour.

The force of the wind, is as the density of the air multiplied into the square of its velocity. SMEATON has given a table of its force, expressed by its pressure in avoirdupois pounds, on a plane of one foot square directly opposed to it. *Phil. Trans.* vol. li. p. 165., &c. Also CAVALLO, vol. ii. p. 287.

A gentle light breeze is there estimated as moving at the rate of four or five miles an hour, and pressing with a force of about two ounces: A brisk but pleasant gale from ten to fifteen miles; with a force from half a pound to a pound: A high wind goes at the rate of thirty or thirty-five miles, and presses with the force of five or six pounds: A hurricane, that tears up trees, and blows down houses, has a velocity of one hundred miles an hour, and a force of forty-nine pounds on a square foot.

The velocity of the wind may be estimated by the velocity of clouds, or of light bodies carried by it.

Rain.

418. The vapour that rises from water uniting itself to the air, ascends into the higher regions of the atmosphere, and is carried by the winds to great distances.

It cannot be doubted, that the humidity raised in this manner is chemically dissolved in the air.

The humidity does not lessen, but increases the transparency of the air, and cannot be withdrawn from it but by substances that attract it powerfully.

419. The power of air to dissolve humidity increases in a greater ratio than that in which its temperature increases.

It appears from SAUSSURE's experiments, that while the temperature increases in arithmetical progression, the humidity which the air is able to contain increases in geometric progression. A cubic foot of air, of the temperature 66°, is able to hold in solution 11 or 12 grains of water: the air itself weighs 570 grains: so that air of the temperature 66°, dissolves about a 50th part of its own weight of water. *Essai sur l'Hygrometrie*, xi. chap. x. p. 167., &c.

420. Hence, if two portions of air, of different temperatures, and both saturated with humidity, be mixed together, a precipitation of humidity must necessarily take place.

If the abscissæ AB, AB′ (fig. 26.) be the temperatures of two equal portions of air, and the ordinates BC, B′C′ represent the quantities of humidity contained in them, the curve CFC′, which represents the dissolving power, corresponding to different temperatures, will be convex toward AB, and will, so far as experiment has shewn, be a logarithmic curve. Now if CC′ be joined, and if BB′ be bisected in D, the perpendicular DE will represent the humidity contained in a given weight of the mixture of the two portions of air, AD the temperature of the mixture, and DF the quantity of humidity which the air of the temperature AD is able to contain. The quantity precipitated from each portion of the air is therefore EF; so that the whole quantity precipitated is 2 EF.

In general, if T and t are the temperatures of two equal portions of air, H and h the humidity contained in them when saturated, the quantity of humidity precipitated from the mixture will be,

$$H + h - 2\sqrt{Hh}.$$

421. If, therefore, large portions of the atmosphere, of different temperatures, and saturated, or nearly saturated, with humidity, be driven against one another by contrary winds, the consequence must be a precipitation of humidity, or the formation of clouds.

This simple and ingenious theory of the formation of clouds, was the discovery of the late Dr HUTTON. See *Edinburgh Transactions*, vol. i. p. 41. It is not necessary to the theory, that the curve which terminates the perpendiculars BC, B′C′, &c. should be a logarithmic curve; it is sufficient that it be a curve convex toward the axis AB. Of this Dr HUTTON had satisfied himself by the observation of natural phenomena. If the line terminating the perpendiculars was straight, and much more were it convex, no precipitation could ever take place.

422. The clouds thus formed, are not disposed equally over the whole atmosphere, but occupy a peculiar region, elevated at an average between two and three miles above the earth.

The mixture of different portions of air is likely to take place most frequently when the two opposite currents already mentioned come in contact with one another. This is at the height of 18,000 feet and upwards, which agrees very well with the medium height of the clouds.

423. The clouds thus formed, have their particles united into larger masses or drops by different causes, such as the mutual attraction of aqueous particles, the force of the wind, or the operation of electricity, and so fall down in rain on the surface of the earth.

The intimate connection between rain and the existence of different currents of air, is evident from many appearances.

1. When the Trade Winds blow uniformly, hardly any rain falls; but when the monsoon changes, heavy falls of rain seldom fail to take place.

2. In the tropical climates, the rainy season is always on the sun's approach to the zenith, at which time also the winds are most variable.

3. There are some spots of continual rain, which seem to be where opposite streams of air constantly meet one another.

4. There are several tracts on the earth's surface, where it hardly ever rains. They are usually far inland, and have extensive plains, without any of

those inequalities of surface that promote the mixture of air.

5. In the midst of those deserts, where mountains occur, moisture is precipitated, sometimes in the form of rain, but most frequently of dew, so that there are springs of fresh water, and great fertility produced.

6. There is in our climate hardly any instance of rain without a change of wind, and very rarely a change of wind without rain in a greater or smaller quantity.

7. The lowness of the mercury in the barometer is a sign of rain. It is a certain indication of the subversion of the equilibrium of the atmosphere, and makes it probable, that before the equilibrium is restored, winds from different quarters, and of different temperatures, must come into collision with one another.

424. The Hygrometer is an instrument intended to measure the quantity of humidity contained in the air at any time. The power of evaporation to produce cold has been very happily applied by Professor LESLIE, to the construction of an instrument of this kind.

425. The vapour raised up into the air, is of a quantity sufficient to afford all the rain that falls,

or to supply all the springs, and of consequence all the rivers derived from them, on the surface of the earth.

Dr Halley shewed, that the evaporation from the sea alone is a sufficient supply for all the waters that the rivers carry into it. His calculation was founded on a very complex view of the subject, and liable to several objections. Buffon took a more simple view of the matter, by selecting one of those lakes that sends out no stream to the ocean, and shewing that the probable evaporation from the surface of the lake is equal to all the water carried into it.

A simpler view still may be taken, by shewing that all the water annually emptied by a river into the sea, is less than the rain that falls on the surface drained by it. Thus, according to Dr Halley's computation, the water which the Thames carries down through Kingston Bridge, to which the tide does not reach, is 25344000 cubic yards, or 684288000 cubic feet *per* day, which gives 249765120000 cubic feet for the quantity of water which the Thames discharges annually into the sea. Now, the surface drained by the Thames and its branches, appears to be about equal to a circle of 40 miles radius; that is, nearly to 5036 square miles, or 140395622400 square feet. But if we suppose that the depth of rain which falls annually, at an average, over this surface is two feet, we have a quantity which exceeds that carried down by the Thames by 31026124800, or nearly an eighth part of the whole, which is therefore left to be taken up by the evaporation.

MARIOTTE has made a similar computation for the Seine, in his *Traité de Mouvement des Eaux.*

426. The disputes that prevailed so long concerning the origin of fountains and rivers, chiefly arose from the difficulty of conceiving, how a precarious and accidental supply could be rendered equal to a regular and great expenditure.

The opinions of the ancients, concerning the origin of fountains, are to be found in SENECA, *Nat. Quæst.* lib. III. See also VARENIUS, sect. iv. chap. xvi. prop. 5.

427. The quantity of rain that falls in different places in the same year, and in the same place in different years, is extremely various; and even in the temperate zone runs between the extremes of 18 and 100 inches.

In places not very distant from one another, the difference in the quantity of rain is often very great. The neighbourhood of the sea on one hand, and mountains on the other, is most favourable to the production of rain; from the first is derived a humid

mid atmosphere, and from the second the prevalence of winds of different temperatures.

When the rain exceeds 25 inches a-year, the climate is to be accounted moist. In the year 1809, the rain that fell at Dalkeith, near Edinburgh, was 28.5; at Glasgow 25.1; at Largs, at the mouth of the Clyde, 40.6; and at Gordon Castle, on the east coast of Scotland, 24.5: At London, in the same year, 20.7; in the preceding year, the fall was but 18.8. In some places of this island, such as Kendal and Keswick, the rain amounts to 64 and 68 inches in a year.

428. The disposition of the rocks in strata, contributes much to the collecting of the waters under the surface, and the conveying them without waste, as if in close pipes, till they are united in fountains, lakes, rivers, &c.

BERNARD PALLASSY was among the first who made this remark, and even brought it to the test of experiment.

429. The region of the air in which the precipitation of humidity takes place, is frequently one where the temperature is below freezing; the frozen particles then uniting, as in the case of melted vapour, form flakes of Snow, which reach the surface in that same state, when the cold of freezing continues down to the surface.

When the aqueous particles first form a drop, and are afterwards frozen in their descent, they become Hail, which is sometimes found crystallised with some degree of regularity. The whiteness and opacity of the hail, is probably owing to the congelation being performed where the air is very rare. Professor LESLIE has remarked, in the curious experiments he has made on the production of ice by evaporation, in a receiver where the air was considerably rarefied, that the ice is more porous, and less transparent, than that which is formed under the ordinary pressure of the atmosphere.

430. Dew is a precipitation of humidity from the lower strata of the atmosphere, which does not disturb the transparency of the air, and to which the mixture of different streams of air does not seem necessary.

Dew is formed only in calm weather, and in the evening, or during the night. The resting of the globules of dew on the points of grass and other plants, seems to indicate the presence of some electrical influence. BRUCE has remarked, that during the dry season no dew falls in Egypt, but that it is usually observed in the Delta five or six days before the inundation.

431. Besides these deposites from the atmosphere, which are all of one substance, there are

others that cannot be traced to the same origin. These are the Stones which have often been said, and have of late years been ascertained beyond all doubt, to fall down from the air; on examination, they are found to contain certain metallic ores, in many respects unlike to any thing that is found either on the surface or in the bowels of the earth.

See an account of Meteoric Stones, HOWARD, *Phil. Trans.* 1802. IZARN, *Lithologie Atmospherique*, Paris, 8vo, 1803: and CAVALLO, vol. iv. p. 372., &c.

In the absence of all analogous appearances, it is perhaps unphilosophical to offer any explanation of the Meteoric Stones. We would only suggest as a mere possibility, that gaseous substances may be thrown up into the air, from the numerous volcanoes on the earth's surface, which may contain the elements of metallic and stony bodies from the mineral regions; and that these, while floating about in the atmosphere, may sometimes be collected in considerable quantities into one place, where, being subjected to electric or galvanic action, they are united into a solid mass.

END OF VOLUME FIRST.

Fig. 1

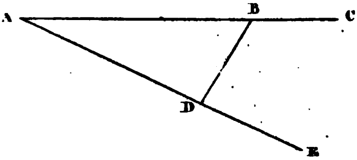


Fig. 2

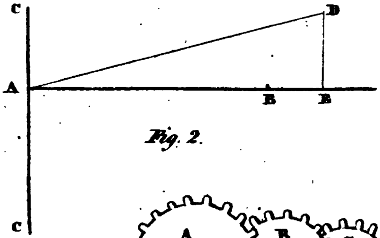
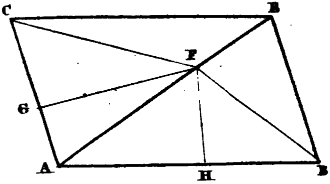


Fig 3



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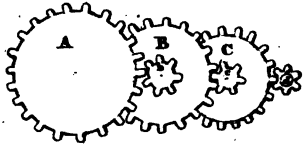

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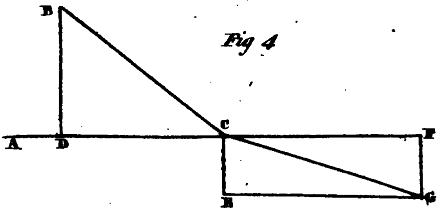

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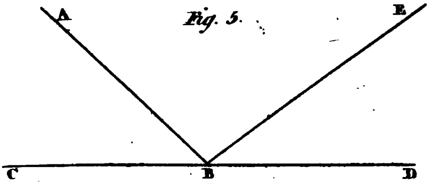

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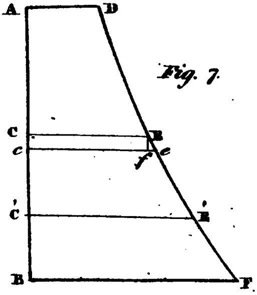

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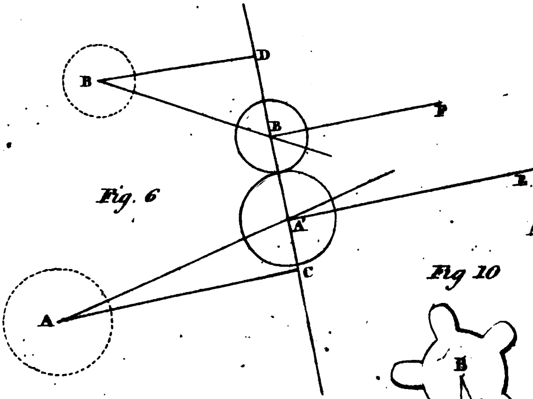

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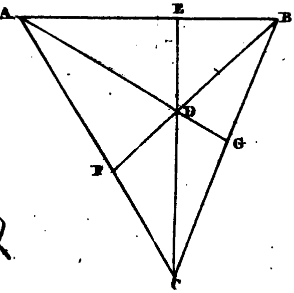
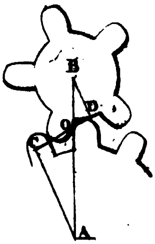
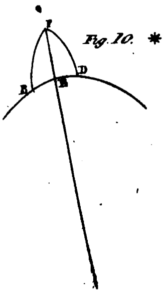


Fig 10





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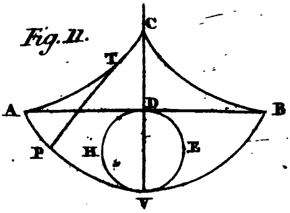

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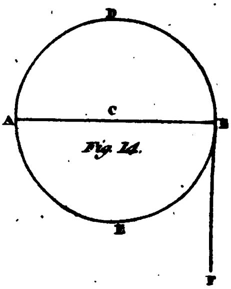

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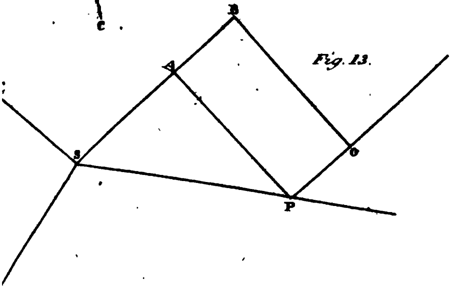

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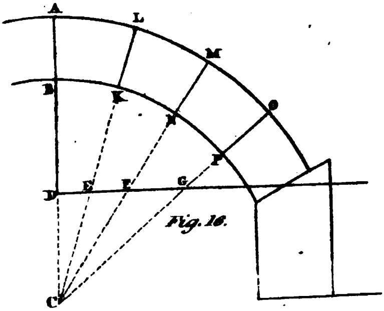

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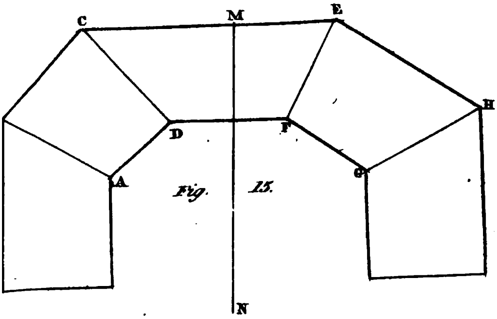

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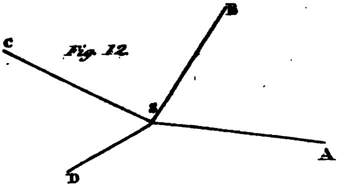

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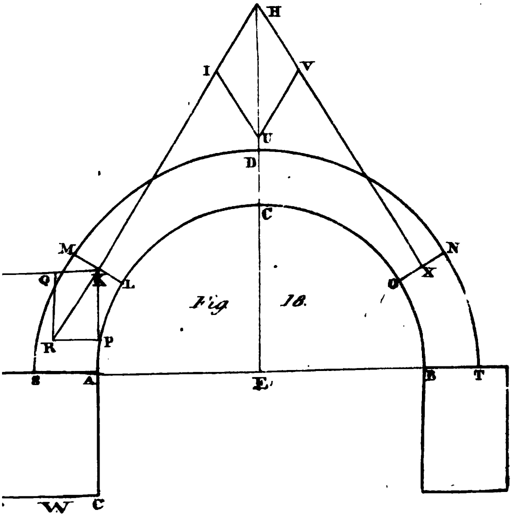

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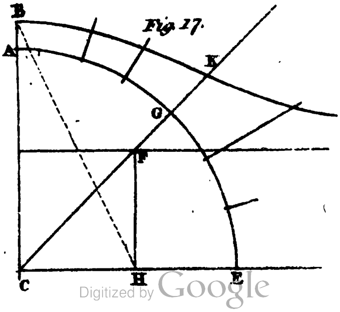

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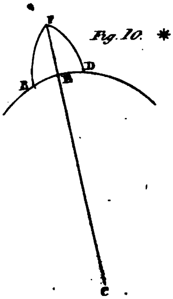

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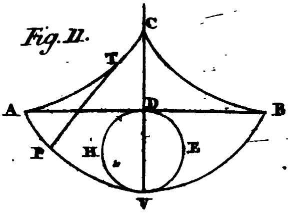

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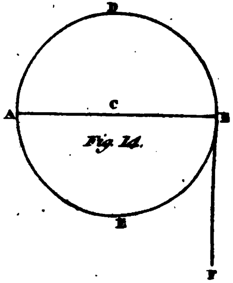

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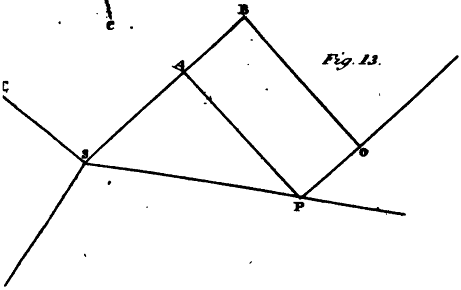

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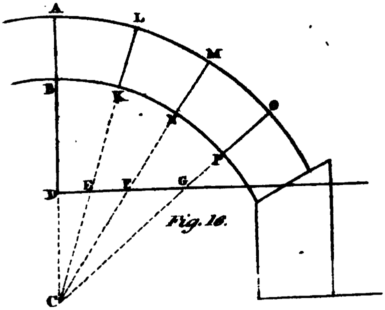

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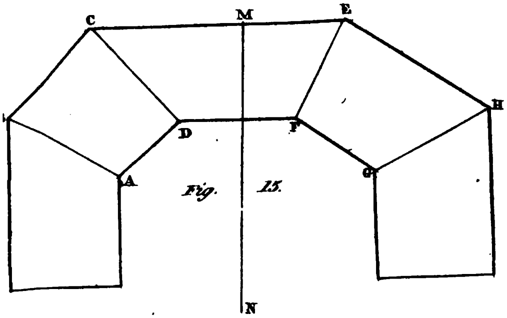

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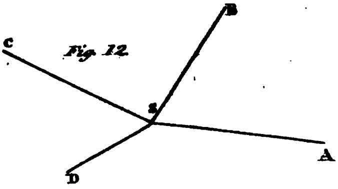

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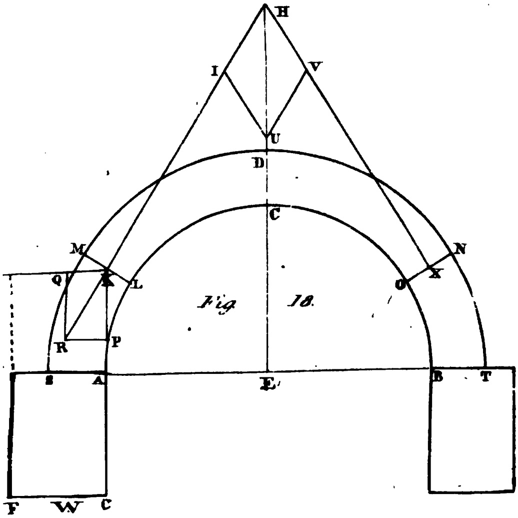

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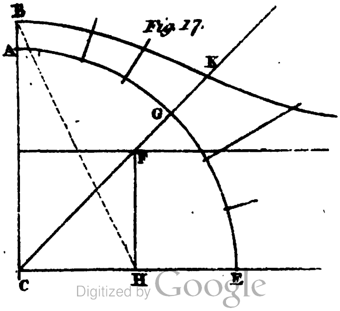

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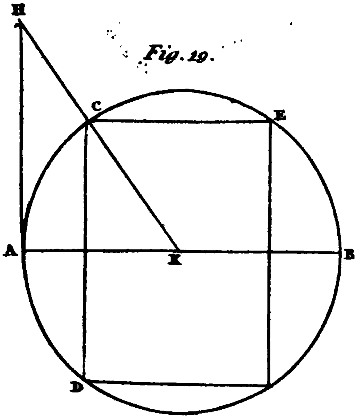

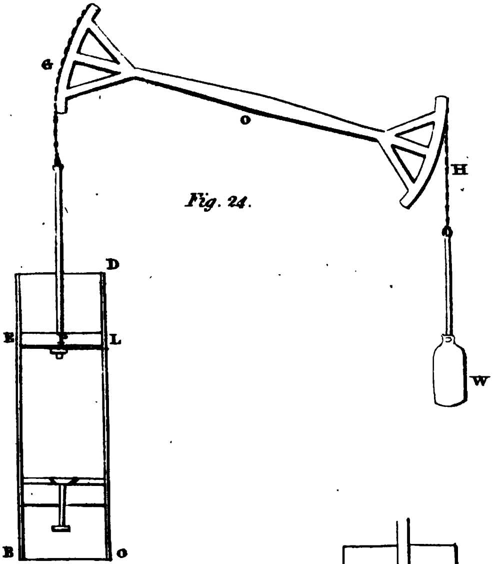

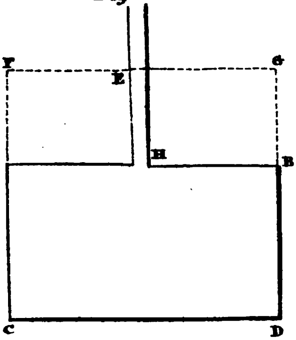
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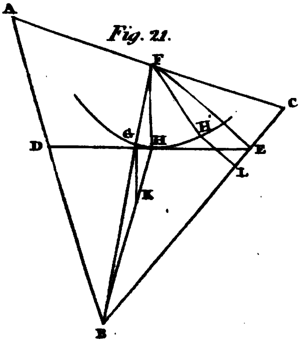

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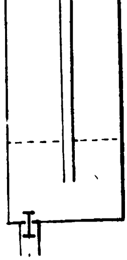

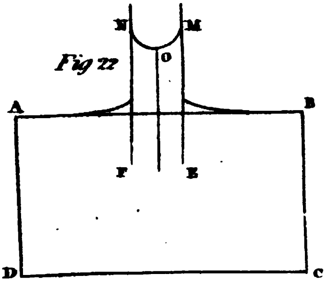

Fig. 25.